The search for Diophantine sextuples and a possible septuple

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Abstract. A rational Diophantine m-tuple is a set of m non-zero rational numbers $\{r_i, r_{i+1}, \ldots, r_m\}$ such that the product of any two plus 1 is a perfect square: $r_i \cdot r_j + 1 = s^2$. Diophantus of Alexandria discussed these numbers in his book IV and gave the first Diophantine quadruple. Leonard Euler proved that an infinite number of Diophantine quintuples exist. In 1999, Phil E. Gibbs found the first example of a rational Diophantine sextuple. Recently a list of 2,001 Diophantine sextuples was archived.

It remains an open question if a Diophantine septuple exists. Finding more sextuples in hope of uncovering the elusive septuple seems to be a perfect fit for a distributed computing project.

Introduction to Diophantine m-tuples

Mathematician Andrej Dujella writes[1]:

The Greek mathematician Diophantus of Alexandria first studied the problem of finding four numbers such that the product of any two of them increased by unity is a perfect square[2]. He found a set of four positive rationals with the this property:

$$1/16, 33/16, 17/4, 105/16 \tag{1}$$

However, the first set of four positive integers with the above property,

$$1, 3, 8, 120 \tag{2}$$

was found by Fermat [3]. Indeed, we have

$$\begin{array}{ll} 1 \cdot 3 + 1 = 2^2 & 1 \cdot 120 + 1 = 11^2 \\ 1 \cdot 8 + 1 = 3^2 & 3 \cdot 120 + 1 = 19^2 \\ 3 \cdot 8 + 1 = 5^2 & 8 \cdot 120 + 1 = 31^2 \end{array}$$

Euler [4] found the infinite family of such Diophantine quadruple sets:

$$a, b, a + b + 2r, 4r(r + a)(r + b) \tag{3}$$

where $ab + 1 = r^2$. He was also able to add the fifth positive rational, 777480/8288641, to Fermat's set. In 2019, Michael Stoll [5] proved that extension of Fermat's set to a rational quintuple with the same property is unique.

In January 1999, the first example of a set of six positive rationals was found by Phil Gibbs [6, 7]:

$$11/192, 35/192, 155/27, 512/27, 1235/48, 180873/16. \tag{4}$$

[end quote from Dujella]

Later on Gibbs provided a list of 644 Diophantine sextuples [8].

It is known now that there exists infinitely many rational Diophantine sextuples, see *"There are infinitely many rational Diophantine sextuples"*, Dujella, et al, etc.[9, 10, 11]. Tito Piezas also posted several formuli giving at least 2 solutions for the sextuple on the mathoverflow.net[12].

2010 *Mathematics Subject Classification.* 11D09, 11G05, 11Y50.

Key words and phrases. Diophantine sextuple.

While these parametric formuli exist, computer searches using the parameters have not found a Diophantine septuple. Furthermore brute force searches have shown that these formuli only provide a small subset of the possible sextuples which do exist over a given range.

In 2024, Randall Rathbun placed a set of 2,001 Diophantine sextuples [13] into the Zenodo archive, many found using exhaustive search techniques, of limited heights.

However, despite computer searches from 1999, uncovering Diophantine sextuples, and parameteric formuli, no such example of a Diophantine septuple is known.

Current Research (2024)

Can we find the elusive Diophantine septuple? 33 pairs of Diophantine sextuples are known which share a common quintuple. For example, the pair:

$$[95/112, 243/560, 1100/63, 1147/5040, 7820/567, 196/45]$$
$$[95/112, 243/560, 1100/63, 1147/5040, 7820/567, 38269/6480]$$

is typical example. Unfortunately here $\frac{196}{45} \cdot \frac{38269}{6480} + 1$ is NOT a square.

Pseudo septuples and octuples are known:

$$\begin{array}{ll} 7 & [16/275, 144/11, 825/16, 561/400, 561/400, 179/1859, 112875/29744] \\ 8 & [99/20, 99/20, 52/165, 52/165, 204/55, 525/44, -2068/25215, 177100/189003] \end{array}$$

but there is a repeated term(s) in the tuple, hence the name pseudo-tuple.

Search Strategies

Phil Gibbs provided several search strategies:

- Step 1: Generate large numbers of rational Diophantine pairs, triples or quadruples of low height.

  For this survey a brute force search for all examples up to a given height was used to form the main base of pairs and triples. It is not hard to generate all triples up to a height of 1000 in this way and a partial search up to a height of 3000 was also used.

- Step 2: Find ways to extend the pairs, triples or quadruples with one additional rational number.

  One method to do this for triples is to resolve the elliptic curve generated by the triple. This is the best technique if many extensions are desired but in practice the larger solutions are unlikely to form sextuples except in special cases that are well understood It is therefore sufficient to extend using simpler methods. This can be done by a combination of extending triples to regular quadruples and quadruples to regular quintuples by well-known known algebraic methods. In addition I give below a parametric solution to extending rational Diophantine pairs to triples that was used extensively in this search.

- Step 3: Scan all the extensions found to see if they can be combined to give a sextuple.

  Checking whether large rational numbers are squares is quite a costly part of the algorithm so it is important to do this only once for each pair and store the results in a Boolean array. This can then be used to quickly check if there are any combinations amongst the extension list which would then complete a rational Diophantine sextuple.

In the mathoverflow.net article[12], Tito Piezas also shows examples of starting with Diophantine triples and possibly extending them to sextuples.

Author's Note: Using isomorphic elliptic curves connected with the 20 Diophantine triples embedded in the Diophantine sextuple, and using adjugate quadruples have proved to be particularly fruitful in discovering sets of sextuples also.

The challenge - find a Diophantine septuple

Can we find a Diophantine septuple? A distributed computing project seems ideal here, to discover more Diophantine sextuples in the hopes of discovering the elusive Diophantine septuple if it exists.

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