International conference
“Diophantine analysis, uniform distributions and applications”
August 25-30, 2003, Minsk, Belarus
Abstracts

Международная конференция
“Диофантов анализ, равномерное распределение и их приложения”
25-30 августа 2003 г., Минск, Беларусь
Тезисы докладов

Минск 2003

ИНСТИТУТ МАТЕМАТИКИ НАН БЕЛАРУСИ
БЕЛОРУССКИЙ ГОСУДАРСТВЕННЫЙ АГРАРНЫЙ ТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ
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BELORUSSIAN STATE AGRARIAN TECHNICAL UNIVERSITY
THE UNIVERSITY OF YORK (ENGLAND)

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В настоящем сборнике представлены тезисы докладов международной конференции "Диофантов анализ, равномерное распределение и их приложения"(DA-2003). Конференция проводится Институтом математики НАН Беларуси, Белорусским государственным аграрным техническим университетом и Йоркским университетом (Англия).

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Contents

ARITHMETIC PROPERTIES OF CERTAIN Q-SERIES.

Amou M. (Japan)
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We consider solutions of certain Q-difference equations and present several results on irrationality and linear independence of their values.

ON A SPECIAL CHARACTER SUM

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Let p be a rational prime, $q = p^s$, $\mathbb{F}_q$ be a finite field of q elements, $a \in \mathbb{F}_q$ and ψ be a nontrivial multiplicative character on $\mathbb{F}_q$. We consider the sum

$$T(\psi) = \sum_{x_1,\dots x_n \in \mathbb{F}_q} \psi(x_1 \cdots x_n)\overline{\psi}(x_1^{m_1} + \cdots + x_n^{m_n} + a),$$

where $m_1, \dots, m_n$ are positive integers such that $m_j \mid (q-1)$ for each $j \in \{1, 2, \dots, n\}$. The sums of this type appear in the problem of finding an explicit formula for the number of solutions of the equation

$$x_1^{m_1} + \cdots + x_n^{m_n} + a = bx_1 \cdots x_n,$$

where $a, b \in \mathbb{F}_q$, $b \neq 0$. Note that the sum $T(\psi)$ is a generalization of the classical Jacobsthal sum.

In this talk we evaluate $T(\psi)$, under the certain restrictions on a, n, q and the exponents.

SOME ADVANCES AND OPEN PROBLEMS IN METRIC THEORY OF DIOPHANTINE APPROXIMATION

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The metric theory of Diophantine approximation began with E. Borel and A. Khintchine in the beginning of the 20th century. In this theory properties of real numbers (or points) are studied from measure theoretic point of view. As it is well known, by Dirichlet's theorem, for any irrational number x there are infinitely many rational numbers p/q satisfying inequality $\left|x - \frac{p}{q}\right| < \frac{1}{q^2}$. And this can not be significantly improved as for the Golden ration $\alpha = (\sqrt{5} - 1)/2$ one has $\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{\sqrt{5}q^2}$ for all $p/q \in \mathbb{Q}$. Now, assuming that we neglect sets of measure zero, Khintchine's theorem provides the complete description of approximation low for almost all real numbers. More precisely, given a positive decreasing *approximation function* $\psi : \mathbb{N} \to \mathbb{R}^+$, for almost every (in Lebesgue measure) real number x the inequality

$$\left|x - \frac{p}{q}\right| < \psi(q)$$

has infinitely many solutions $p/q \in \mathbb{Q}$ if the sum

$$\sum_{h=1}^{\infty} \psi(q)$$

diverges and it has finitely many solutions if that sum converges.

In higher dimensional spaces Diophantine approximation take various forms such as simultaneous and linear (or dual), standard and multiplicative.

The general idea of metric Diophantine approximation is to take the set of points satisfying some approximating properties and then to investigate measure theoretic properties of this set such as Lebesgue measure, Hausdorff measure and dimension.

In this talk some of the history, problems and recent advances regarding metric theory of Diophantine approximation will be discussed. The following two references are also suggested for further reading.

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HAUSDORFF DIMENSION IN DIOPHANTINE APPROXIMATION ON MANIFOLDS

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Measure theory in Diophantine approximation was used for the first time by O. Borel and A. Khintchine. A little bit later V. Jarnik and A. Besikovitch used Hausdorff dimension. B.Volkmann, V.G. Sprinzuk, R. Baker and the author developed methods for upper estimations of Hausdorff dimension, A. Baker, V. Schmidt, M. Dodson and V.V. Beresnevich worked out the methods for lower estimation based on a fruitful idea of regular systems.

The author will analyze methods mentioned above, and talk about the latest results on upper estimation in the problems of Diophantine approximations on manifolds.

ON EVALUATION OF A SUM OF PRODUCTS OF INVERSE DISTANCES TO THE NEAREST INTEGER

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Consider the problem of a unique distribution of fractional parts in the sequence $\{k\alpha\}_{k=1}^{\infty}$. One of the solutions of this problem requires an evaluation of the sums $\sum_{k=1}^{K}\|k\alpha\|^{-1}$ (for the details refer to [1]). A similar question will arise when investigating the problem of a unique distribution of fractional parts of the linear combinations $\{\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_n a_n\}$. In that case, one has to evaluate the sums $\sum_{k=1}^{K}\prod_{j=1}^{n}\|k\alpha_j\|^{-1}$. Placing the following restriction on the vector $\overline{\alpha}=(\alpha_1,\cdots,\alpha_n)$

$$\forall\epsilon>0\,\exists k_0=k_0(\overline{\alpha},\epsilon)\ ,\ \forall k>k_0\prod_{j=1}^{n}\|k\alpha_j\|^{-1}>k^{-1-\epsilon}. \tag{1}$$

the following asymptotic equality is proved.

Theorem 1. *Suppose that a vector $\overline{\alpha}\in\mathbb{R}^n$ satisfies (1). Then we have*

$$\sum_{k=1}^{K}\prod_{j=1}^{n}\|k\alpha_j\|^{-1}=O(K^{1+\epsilon_2}),$$

where $\epsilon_2\to 0$ as $\epsilon\to 0$.

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ON EXACT ORDER OF SIMULTANEOUS APPROXIMATION OF ZERO IN $\mathbb{R}^3$.

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The direction of studies formulated in the title was indicated by the works of V. G. Sprindzhuk [1,2] in which he posed the basic problem of this direction, solved particular cases of the problem, and pointed out some applications. The conjecture of V. G. Sprindzhuk is as follows. Suppose that $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a polynomial with integer coefficients, $H = H(P) = \max_{0\le i\le n} |a_i|$ is the height of $P(x)$.

Let $\nu_n(\overline{\omega})$ be the least upper bound $\nu > 0$, for which the system of inequalities

$$\max(|P(\omega_1)|, ..., |P(\omega_k)|) < H^{-\nu}$$

has an infinite number of solutions in polynomials $P(x) \in Z[x]$. Is it true that for almost all $\overline{\omega} = (\omega_1, ..., \omega_k)$

$$\nu_n(\overline{\omega}) = \frac{n+1}{k} - 1 \text{ ?}$$

The conjecture of V. G. Sprindzhuk in a more generalized form was proved by V. I. Bernik [3]. Let $w_n(\overline{\omega})$ be the least upper bound $w > 0$, for which the inequality

$$\prod_{i=1}^{k} |P(\omega_i)| \le H^{-w} \tag{1}$$

has an infinite number of solutions in polynomials $P(x) \in Z[x]$.

Then for almost all $\overline{\omega} : w_n = n - k + 1$. Later various generalizations and applications of the result were obtained [4,5]. The work [6] lets to pass from power function to any function $\Psi(x)$ in the second member of the inequation (1), which monotonically decreases for $x > 0$, and $\sum_{H=1}^{\infty} \Psi(H) < \infty$.

We obtain three-dimensional analog of the theorem in [6]; this analog can be considered as a proof of a three-dimensional generalization of the Baker's conjecture.

Theorem 1. *Let the function $\Psi(x)$ monotonically decrease for $x > 0$, and $\sum_{H=1}^{\infty} \Psi(H) < \infty$. The system of inequalities*

$$\begin{cases} |P(\omega_1)| < H^{-w_1}\Psi^{\nu_1}(H) \\ |P(\omega_2)| < H^{-w_2}\Psi^{\nu_2}(H) \\ |P(\omega_3)| < H^{-w_3}\Psi^{\nu_3}(H) \end{cases} \tag{2}$$

where $w_1 + w_2 + w_3 = n - 3, \nu_1 + \nu_2 + \nu_3 = 1$, has only a finite number of solutions in polynomials $P(x) \in Z[x]$ for almost all $(\omega_1, \omega_2, \omega_3) \in R^3$.

The proof is based on the essential and inessential domains method by V. G. Sprindzhuk [2].

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DIOPHANTINE DESCRIPTION OF SEQUENCES OF PRIME NUMBERS

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The solvability over the ring of integers $\mathbb{Z}$ of some diophantine equations is connected with the ability of numbers to form the sequences of prime numbers, in particular with the quality of numbers to be prime twins.

Theorem 1. *A positive integer* $n \equiv 1 \pmod 8$ *is the first prime twin, i.e.* n *and* $n+2$ *are prime numbers, if and only if:*

1. $n = \frac{2x_1^2+x_2^2}{2x_1+1}$ $(x_i \in Z)$,
2. x_1 *and* x_2 *are relatively prime numbers,*
3. *the number of representations* (x_1, x_2) *of* n *is* 8,
4. $n,\ n+2$ *are squarefree numbers.*

This theorem is the natural continuation of the Fermat theorem about two squares, according to which any prime number $p \equiv 1 \pmod 4$ is represented by two squares $p = x^2 + y^2$ and the number of such representations is 8. The present theorem allows to obtain the same characteristic of the first prime twin p among the pair of prime numbers $p,\ p+2$: such number is presented as a rational fraction of a special kind $p = (2x_1^2+x_2^2)/(2x_1+1)$, and the number of such representations is again 8 .

The results obtained for the prime twins are extended onto sequence $k = 4, 6, 8, 10, 12$ prime numbers. For indicated sequences of prime numbers it is proved that the least prime from k numbers is an integer root for some family of polynomials of degree not exceeding k, i.e. it's shown that such prime numbers admit algebraic parametric representation by the roots of polynomials with integer coefficients.

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ALGEBRAIC INDEPENDENCE OVER $\mathbb{Q}_p$

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We report on joint work with Kumiko Nishioka. Let $f(x)$ be a power series $\sum_{n\geq 1}\zeta(n)x^{e(n)}$, where $(e(n))$ is a strictly increasing linear recurrence sequence of non-negative integers, and $(\zeta(n))$ a sequence of roots of unity in $\overline{\mathbb{Q}}_p$, the algebraic closure of $\mathbb{Q}_p$, satisfying an appropriate technical condition. Then we are mainly interested in characterizing the algebraic independence over $\mathbb{Q}_p$ of elements $f(\alpha_1),\ldots,f(\alpha_t)$ from $\mathbb{C}_p$ in terms of the distinct $\alpha_1,\ldots,\alpha_t\in\mathbb{Q}_p$ satisfying $0<|\alpha_\tau|_p<1$ for $\tau=1,\ldots,t$. A striking application of our basic result says that, in the particular case $e(n)=n$, the set $\{f(\alpha)\mid \alpha\in\mathbb{Q}_p, 0<|\alpha|_p<1\}$ is algebraically independent over $\mathbb{Q}_p$ if $(\zeta(n))$ satisfies the "technical condition". We shall end the talk by stating a conjecture concerning more general sequences $(e(n))$.

DIOPHANTINE APPROXIMATION OVER THE REAL AND COMPLEX NUMBERS

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VARIATIONS WITH MAHLER'S MEASURE

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Let α be an algebraic number of degree d over $\mathbf{Q}$ with minimal polynomial

$$a_d z^d+\cdots+a_1z+a_0=a_d(z-\alpha_1)\ldots(z-\alpha_d)\in\mathbf{Z}[z].$$

Its *Mahler measure* is defined by $M(\alpha)=a_d\prod_{j=1}^d\max\{1,|\alpha_j|\}$. It is well–known that, for every $\alpha\in\overline{\mathbf{Q}}$, $M(\alpha)$ is a real algebraic integer greater than or equal to 1.

Let $\mathcal{M}$ be the set of all Mahler measures of algebraic numbers, and let $\mathcal{M}^*$ be a monoid under multiplication generated by $\mathcal{M}$. By the multiplicative property of Mahler measures $\mathcal{M}^*$ is the set of all Mahler measures of integer (not necessarily irreducible) polynomials. The task of thorough investigation of the sets $\mathcal{M}$ and $\mathcal{M}^*$ is considered to be a very ambitious one, since even simply looking Lehmer's question [5] on whether there are elements of $\mathcal{M}$ in the interval $(1, 1.176)$ remains open.

In [1] D.W. Boyd found a necessary geometric condition for a number to belong to $\mathcal{M}$. Every $\alpha\in\mathcal{M}$ must be an algebraic integer having its other conjugates in the annulus $\alpha^{-1}\leq|z|<\alpha$. This however leaves open the possibility for numbers like $\alpha_0:=1.19385\ldots$ solving $z^5-z^2-1=0$ to belong to $\mathcal{M}$. In [2] we prove a result which shows that, in principle, the problem of determining whether any specific α belongs to $\mathcal{M}^*$ or not can be solved. Its partial case can be stated as follows.

Theorem 1. *Suppose that α is an algebraic number of degree d, and F is the Galois closure of $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$. If $\alpha\in\mathcal{M}^*$ then $\alpha=M(f)$ for some separable polynomial $f(z)\in\mathbf{Z}[z]$ of degree at most 2^d whose roots lie in F.*

In particular, the theorem implies that $\alpha_0\notin\mathcal{M}^*$. In [2] we also show that $\mathcal{M}\neq\mathcal{M}^*$, by giving an explicit example.

I will also sketch some of the results of [3] and [4] concerning various problems connected with Mahler's measure.

This research was partially supported the Lithuanian State Science and Studies Foundation.

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SAILS AND NORM MINIMA OF LATTICES

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If $\Lambda \subset \mathbb{R}^n$ is an n–dimensional lattice, then its norm minimum $N(\Lambda)$ is defined as

$$\inf_{\mathbf{x}\in\Lambda\setminus\{0\}} |x_1 \cdot \ldots \cdot x_n|.$$

Consider an irrational n–dimensional lattice $\Lambda \subset \mathbb{R}^n$. The convex hull K of the set of lattice points with positive coordinates is called a Klein polyhedron. Its boundary ∂K is called a sail. Similarly we can define a sail in each of the 2^n orthants and thus obtain 2^n sails generated by Λ. The described construction is one of the most natural multidimensional geometric generalizations of continued fractions. The role of partial quotients is played by the determinants of the sails' faces. (By a determinant of face F we mean the volume of the pyramid $\mathrm{conv}(F \cup \{0\})$ multiplied by $n!$).

It is well known that a number α is badly approximable (i.e. there exists such a constant c that for each convergent p/q of α $|q\alpha - p| > c/q$) if and only if the partial quotients of α are bounded. We proved an analogous statement for sails:

Theorem 1. *Let Λ be an irrational n–dimensional lattice in $\mathbb{R}^n$ with $\det\Lambda = 1$. Then $N(\Lambda) = \mu > 0$ if and only if there exists such a constant D that for each face F of each of the 2^n sails generated by Λ we have $\det F < D$.*

Horeover, D depends only on μ and does not depend on Λ, same as μ depends only on D.

ON MOTIVES OF SCHEMES OF DIMENSION ONE

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Let C be a $\mathbb{Q}$-linear, pseudoabelian and symmetric monoidal category with a product $\otimes$. Let n be a natural number and let Σ_n be the symmetric group of permutations of n elements. For any $X \in \mathsf{C}$ one can define its wedge $X^{[n)}$ and symmetric $X^{(n]}$ powers as images of the idempotents in $End_{\mathsf{C}}(X^{\otimes^n})$ corresponding to the "vertical"and "horizontal"irreducible representations of Σ_n over $\mathbb{Q}$. These powers generalize usual wedge and symmetric powers of vector spaces over a field of characteristic zero. Then X is called to be evenly (oddly) finite dimensional if $X^{[n)}$ (or, respectively $X^{(n]}$) is a zero object for some n. X is said to be finite dimensional if $X \cong X_+ \oplus X_-$ where X_+ is evenly and X_- is oddly finite dimensional.

The theory of finite dimensional Chow motives was introduced by S.-I. Kimura in [6], and then considered in [4] and [5]. The abstract theory was developed independently by O'Sullivan, [1] (compare with the concept of a Schur functor in [2]).

Let k be a field and let $\mathsf{DM}^-(\mathsf{k})_{\mathbb{Q}}$ be the $\mathbb{Q}$-localized Voevodsky's triangulated category of motives over k, [8]. The following theorem generalizes motivic finite dimensionality for smooth projective curves proved in [6] to arbitrary schemes of dimension one:

Theorem 1. *Let k be a field of characteristic zero and let X be an integral scheme of dimension one, separated an of finite type over k. Then its motive $M(X)$, considered in Voevodsky's category $\mathsf{DM}^-(\mathsf{k})_{\mathbb{Q}}$, is finite dimensional.*

This result is a corollary of the additivity for Kimura finite vertices in distinguished triangles in quite general triangulated categories represented as homotopy categories of pointed model and monoidal categories with invertible simplicial suspension, see [3].

The same result as in the theorem above has been independently obtained by Carlo Mazza, [7].

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ON LINEAR COMBINATIONS OF LOGARITHMS OF ALGEBRAIC NUMBERS WITH ALGEBRAIC COEFFICIENTS

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Let $\{d,m,n\}\subset\mathbb{N}$, $d\geq 2$, $K=\mathbb{Q}[\exp(2\pi i/m)], \Lambda(n)$ be the Mangold's function, $T\in\mathbb{R}$,

$$\varepsilon^2=1,\ \Lambda_0(m)=(1-2\{m/2\})\Lambda(m/(2-2\{m/2\})),$$

$$\omega(m)=4(1-\{(m+2)/4\})(1-2\{m/2\})+2\{m/2\},$$

$\log(z)$, where $z\in\mathbb{C}\diagdown(-\infty,0]$ is a branch of the logarithm with $|\arg(z)|<\pi$. Further, let

$$w_d(T)>0,\ 2(w_d(T))^2=((d^2(3-T^2)+1)^2+16d^4T^2)^{1/2}+d^2(3-T^2)+1,$$

$$V_{d,k}(m)=k(d+1)\Lambda_0(m)/\phi(m)\ +\ln((d-1)^{(d-1)/2}(d+1)^{(d+1)/2}d^{-d})+$$

$$(\pi/2)\sum_{\mu=0}^{1}(1-2\mu)\sum_{\kappa=1}^{[(d-1)/2]+\mu}\cot\left(\frac{\pi\kappa}{d-1-2\mu}\right),$$

$$l_d(\varepsilon,T)=-\ln(4(d+1)^{d+1}(1-1/d)^{d-1})+$$

$$\sum_{k=-1}^{1}((d-1)^{1-|k|}/2)\ln((2d+(d+1)k+\varepsilon w_d(T))^2+(dT(1+2d\varepsilon/w_d(T))^2),$$

$$g_{d,k}(m)=(-1)^k l_d((-1)^k,\cot(\pi\omega(m)/(2m))+V_{d,1}(m)),$$

$$h_{d,k}(m)=-V_{d,k}(m)-l_d(1,\tan(\pi/m)),$$

where $m\neq 2$, $k=0,1$. Let

$$\beta(d,m)=g_{d,0}(m)/h_{d,1}(m),\ \alpha(d,m)=\beta(d,m)-1+g_{d,1}(m)/h_{d,1}(m).$$

Theorem 1. *If $m \in \mathbb{N}\diagdown\{1,2,6\}$, then $0 < h_{d,1}(m)$ and for each $\varepsilon > 0$ there exists $C_{d,m}(\varepsilon) > 0$ such that*

$$\max_{\sigma\in Gal(K/\mathbb{Q})}(|q^\sigma \log((2+\exp(2\pi i/m))^\sigma) - p^\sigma|) \geq$$

$$\geq C_{d,m}(\varepsilon)(\max_{\sigma\in Gal(K/\mathbb{Q})}(|q^\sigma|)^{-\alpha(d,m)-\varepsilon},$$

where $p \in \mathbb{Z}_K$, $q \in \mathbb{Z}_K\diagdown\{0\}$; furthermore, for any $q \in \mathbb{Z}_K\diagdown\{0\}$ and any $\varepsilon > 0$ there exists $C^*_{d,m}(q,\varepsilon) > 0$ such that

$$b^{\beta(d,m)+\varepsilon}\max_{\sigma\in Gal(K/\mathbb{Q})}(|q^\sigma b\log((2+\exp(2\pi i/m))^\sigma) - p^\sigma|) \geq C^*_{d,m}(q,\varepsilon),$$

where $p \in \mathbb{Z}_K$, $b \in \mathbb{N}$.

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PERFECT POWERS FROM PRODUCTS OF CONSECUTIVE TERMS IN ARITHMETIC PROGRESSION

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The study of the diophantine equation

$$n(n+d)\dots(n+(k-1)d) = by^l \tag{1}$$

in integers $n, d, y, b \geq 1$, $k, l \geq 2$ with $\gcd(n,d) = 1$, $P(b) \leq k$ goes back to the 17th century. There is an extremely rich literature of equations of this form.

In the classical case $d = 1$ it was proved by Erdős and Selfridge that (1) has no solution for $b = 1$. A similar result was proved by Erdős ($k \geq 4$) and Győry ($k = 2,3$) for $b = k!$ (binomial equation). A common generalization of these was established by Saradha ($k \geq 4$) and Győry ($k = 2,3$) for any b with $P(b) \leq k$.

The general case $d \geq 1$ is more complicated. Although equation (1) has attracted the attention of many mathematicians, only some partial results are known so far. In our talk some recent results will be presented which have been obtained jointly with M. Bennett, L. Hajdu and N. Saradha.

Together with Hajdu and Saradha we proved that for fixed $k \geq 3$ and $l \geq 2$ with $k + l > 6$, equation (1) has only finitely many solutions in n, d, b, y. Moreover, we deduced from the ABC-conjecture that under the conditions $d > 1$, $k \geq 3$ and $l \geq 4$, (1) has only finitely many solutions in n, d, k, b, y, l. Jointly with Bennett and Hajdu we showed that if $k \geq 4$ is fixed, then under certain technical assumptions (1) has at most finitely many solutions in n, d, b, y, l with $P(b) < k/2$.

For small values of k, we established a more precise result. Namely, we showed that for $3 \leq k \leq 11$, the product of k consecutive terms from a positive arithmetic progression is never a perfect power. For $k = 3$ this is due to Győry, for $k = 4,5$ to Győry, Hajdu and Saradha and for $6 \leq k \leq 11$ to Bennett, Győry and Hajdu, respectively. In fact we proved a more general version of this theorem when n and b are non-zero integers and $P(b) < \max\{3, k/2\}$.

To prove our results, we reduced first equation (1) to ternary equations of the form

$$Ax^l + By^l = Cz^l \quad \text{or} \quad Cz^2.$$

Then we combined several classical and modern results and methods, including Frey curves, Galois representations and modular forms, to solve the above ternary equations.

THE INTEGER POINTS CLOSE TO A PLANE CURVE AND RELATED PROBLEMS

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Some problems in number theory can be interpreted as the distribution of a certain set of points with integer coordinates. We use trigonometric sums for counting the number of integer points inside a closed plane curve. Some problems require an estimate for the number of integer points in a narrow strip alongside a plane curve. If the strip is wide, trigonometric sum methods are still useful, and they give an approximate formula. For a narrow strip we cannot expect an asymptotic formula, so upper and lower bounds are of interest. Lower bounds, where possible, are by constructions. Upper bounds come from spacing ideas. The simplest spacing idea is that if there are more than $2A+2$ integer points in a convex set area A, then they are all on a straight line. Either there are at most $2A+2$ integer points (a 'minor arc' spacing property) or the integer points in the region satisfy some algebraic equation (a 'major arc'). More complicated arguments of this type use interpolation determinants and differential inequalities. The 'depth' is only that of Liouville's approximation theorem, but working in two dimensions ensures considerable complications.

SMALL DENOMINATORS FOR THE DIRICHLET PROBLEM IN A DISK

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In the unit disk $K=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$ the Dirichlet problem is considered: to find the solution of partial differential equation with constant complex coefficients a_j

$$L\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)u\equiv\sum_{j=0}^{2m}a_j\frac{\partial^{2m}u}{\partial x^{2m-j}\partial y^j}=0, \tag{1}$$

which satisfies on the disk's boundary ∂K the Dirichlet conditions

$$\frac{\partial^j u}{\partial\nu^j}(x,y)=\varphi_j(x,y),\qquad j=0,1,\dots,m-1, \tag{2}$$

where $\varphi_0(x,y)$, $\varphi_1(x,y)$, $\dots$, $\varphi_{m-1}(x,y)$ are known functions and ν is an outside normal on ∂K.

The solvability conditions of this problem are established by V. P. Burskii (Investigation methods of boundary value problems for general differential equations, Kyiv, Naukova dumka, 2002, 315 p.) and by E. A. Burjachenko.

In particular, if the roots λ_j of polynomial $L(1,\lambda)$ are simple and not equal to $\pm i$, then $\det\delta_n\neq 0$ for all integer $n\geq 2m$, where matrix

$$\delta_n=\begin{pmatrix}\cos n\varphi_1 & \sin n\varphi_1 & \dots & \cos(n-2m+2)\varphi_1 & \sin(n-2m+2)\varphi_1\\ \cos n\varphi_2 & \sin n\varphi_2 & \dots & \cos(n-2m+2)\varphi_2 & \sin(n-2m+2)\varphi_2\\ \dots & \dots & \dots & \dots & \dots\\ \cos n\varphi_{2m} & \sin n\varphi_{2m} & \dots & \cos(n-2m+2)\varphi_{2m} & \sin(n-2m+2)\varphi_{2m}\end{pmatrix}$$

and complex angle φ_j satisfies the equation $\tan\varphi_j=-\lambda_j$, is a necessary and sufficient condition for uniqueness of the solution of the problem (1), (2). Analogous conditions are established in the case when λ_j is a multiple root and/or $\pm i$ is a root of the polynomial $L(1,\lambda)$.

For $m=1$ the uniqueness criterion of the solution of the problem (1), (2) is π-irrationality of the number $\varphi_2-\varphi_1$. For $m>1$ π-irrationality of at least one of the numbers $\varphi_2-\varphi_1$, $\varphi_3-\varphi_1$, $\dots$, $\varphi_{2m}-\varphi_1$ is the necessary uniqueness condition. Non-uniqueness condition can be formulated in such way: there exists a nontrivial polynomial $h(\lambda)$ Of the degree not higher than $n-2m$ and polynomials $h_+(\lambda)$ and $h_-(\lambda)$ of the degree not higher than $m-1$, for which the identity $L(1,\lambda)h(\lambda)+(\lambda+i)^{n-m+1}h_+(\lambda)+(\lambda-i)^{n-m+1}h_-(\lambda)\equiv 0$ holds.

If S_n is a matrix with the i-th column formed by the coefficients of the polynomial $L(1,\lambda)\lambda^{n-2m+1-i}$, and $S_{n\pm}$ is a matrix with the i-th column formed by the coefficients of the polynomial

$(\lambda \pm i)^{n-m+1}\lambda^{m-i}$, then the uniqueness condition consists in nondegeneration of the matrix (S_n, S_{n+}, S_{n-}) for all integer $n \geq 2m$.

To obtain the existence of a solution of the problem (1), (2) in the scales of functional spaces it is sufficient to estimate the norm of reciprocal matrix δ_n^{-1}. For this we have to estimate the determinant of matrix δ_n, which has $\Delta_n = \det(S_n, S_{n+}, S_{n-})$ as its multiplier. Numbers Δ_n can take as small values as desired for infinite number set n. The problem of estimating of this denominators consists in proving the lower estimates $|\Delta_n| \geq C\rho_n$ for almost all coefficients of the polynomial $L(1,\lambda)$, where the constant C is independent from n, and function ρ_n is independent from the coefficients of the polynomial $L(1,\lambda)$.

BENFORD'S LAW AND RATIONAL APPROXIMATIONS OF LOGARITHMS OF REAL NUMBERS

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If the initial zeros are skipped, the decimal representation of any number begins with one of the nine digits, $a = 1, 2, \ldots, 9$. For most statistical data, the distribution of those digits isn't uniform, but instead each digit a occurs with a frequency close to

$$\lg \frac{a+1}{a}. \tag{1}$$

This observation is due to Simon Newcomb [1], an astronomer and mathematician. Later, Benford published the results of a more in-depth study [2], giving a number of real-life examples where the distribution defined by (1) is found. Thus, it became know as Benford's law.

Let $\{x\}$ denote the fractional part of a real number x. Consider the sequence 2^n, $n = 1, 2, \ldots$. The distribution of the first digits in it depends on the distribution of the numbers $\{n \lg 2\}$, which is uniform since the number $\lg 2$ is irrational. Thus, it follows Benford's law. Moreover, the divergence between the actual proportion of different digits and the values given by (1) depends on the measure of irrationality of $\lg 2$ and it can be estimated as follows.

Lemma 1. *Let*

$$\left|\alpha - \frac{p}{q}\right| > c(\alpha)q^{-\lambda}, \ \lambda \geq 2. \tag{2}$$

hold for some irrational α and all integers $q > 0$ and p. Then

$$\sum_{\nu=1}^{L} \frac{1}{\|\nu\alpha\|} \leq c_1(\alpha)L^{\lambda-1}\ln L, \tag{3}$$

holds for any integer $L > 8$ with $c_1 = \frac{2^{\lambda+1}}{c(\alpha)}$.

Thus we obtain the following theorem:

Theorem 1. *Let $B(A,Q)$ be a number of integers n, $1 \leq n \leq Q$, such that the decimal representation of 2^n begins with A. For all $\varepsilon > 0$ the following asymptotic estimate is true*

$$B(A,Q) = Q \lg \frac{A+1}{A} + O(Q^{1-\frac{1}{\lambda_0 - 1}+\varepsilon}), \tag{4}$$

where $\lambda_0 \geq 2$ is a positive real number such that the inequality

$$\left|\lg 2 - \frac{p}{q}\right| > cq^{-\lambda_0}, \ \lambda \geq 2$$

holds for some positive constant c and all integers $q > 0$ and p.

This theorem can be immediately translated into a result for 2^n by using the known fact that λ_0 can be taken to be equal to 2^{42}.

Theorem 2. *Let p_k be the k-th prime; for any positive integers $A_1, A_2, \dots, A_k$ let $B(A_1, \dots, A_k, Q)$ be the number of integers n, $1 \le n \le Q$, such that the decimal representation of 2^n begins with A_1, the decimal representation of 3^n begins with A_2, etc. There exists a μ_1, $0 < \mu_1 < 1$, such that for all ε_1, $0 < \varepsilon_1 < 1 - \mu_1$ and $Q \to \infty$ the following asymptotic expression is true*

$$B(A_1, \dots, A_k, Q) = Q \prod_{s=1}^{k} \lg \frac{A_s + 1}{A_s} + O_{\varepsilon_1}(Q^{\mu_1 + \varepsilon_1}) \tag{5}$$

It is obvious that the numbers $p_1, \dots, p_k$ don't have to be primes, and the linear independence of their decimal logarithms is the necessary and sufficient condition for (5).

Finally, it is possible to further extend the problem by considering the decimal representation of the numbers in the sequence from $(s+1)$-th position, counting from the beginning. The we have

Theorem 3. *For $Q \to \infty$ we have*

$$B_2^{(s)}(Q) = Q\nu_s(A_1) + O_\varepsilon(Q^{1 - \frac{1}{\lambda_0 - 1} + \varepsilon}), \tag{6}$$

where the constant which is implicitly present in the Vinogradov symbol is $9 \cdot 10^{s-1}$ bigger then one in Theorem 1.

However, experimental evidence shows that this constant can be improved, which is supported by the following lemma.

Lemma 2. *Consider irrational β such that the inequality (2) holds for all $(p, q) \in \mathbb{Z} \times \mathbb{N}$. For all integer $M > 1$, $S \le M$, and all real a and b, $0 < b - a < M^{-1}$ and*

$$V = \bigcup_{j=0}^{S} \left[a + jM^{-1}, b + jM^{-1}\right)$$

we have

$$N_V(\beta, Q) = |V|\, Q + O_\varepsilon\left(Q^{1 - \frac{1}{\lambda - 1}} \ln Q\right),$$

where the constant in the Vinogradov symbol is not larger than

$$2^{2\lambda + 10} \ln M \left(\frac{1}{c(M\beta) M^{\lambda - 2}} + \frac{1}{c(\beta)}\right)$$

.

Finally, we can replace the natural number a by any real number. Let

$$\lg a_1 = \frac{\sqrt{5} - 1}{2}, \ \lg a_2 = \sqrt{2}, \ \lg a_3 = e, \lg a_4 = \pi$$

and let $B_{a_j}(Q)$, $1 \le j \le 4$ be the number of integer n, $1 \le n \le Q$, such that a_j^n begins with a natural number A. Using known results on the approximations of $\lg a_j$, we obtain the following theorem.

Theorem 4. *When $Q \to \infty$ we have*

$$B_{a_1}(Q) = Q \lg \frac{A + 1}{A} + O(\ln Q) \tag{7}$$

$$B_{a_2}(Q) = Q \lg \frac{A + 1}{A} + O(\ln Q) \tag{8}$$

$$B_{a_3}(Q) = Q \lg \frac{A + 1}{A} + O(\ln^2 Q) \tag{9}$$

$$B_{a_4}(Q) = Q \lg \frac{A + 1}{A} + O(Q^{\frac{5}{6}}) \tag{10}$$

For almost all a and any $\varepsilon > 0$ we have

$$B_a(A, Q) = Q \lg \frac{A + 1}{A} + O(\ln^{2+\varepsilon} Q) \tag{11}$$

Take $m_1 = 10^x, \dots, m_k = 10^{x^k}$. By using a metrical estimate of the values of integer polynomials [8] and taking $B'(A_1, \dots, A_k, Q)$ to be equal $B(A_1, \dots, A_k, Q)$, similarly to theorem 4, with p_j replaced by m_j, we obtain

Theorem 5. *For almost all x and any $\delta > 0$ we have*

$$B'(A_1, \dots, A_k, Q) = Q \prod_{s=1}^{k} \lg \frac{A_s + 1}{A_s} + O_\delta(\ln^{k+\delta} Q)$$

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THE DISTRIBUTION OF RATIONAL POINTS NEAR a HYPERBOLIC PARABOLOID

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The question of the distribution of rational points in domains of n-dimensional Euclidean spaces is one of the central problems of the number theory. Just problems connected with the distribution of rational points in domains where one of dimensions approximates to zero as soon as denominator of rational numbers increases are very popular now owing to numerous applications

At present exact upper and lower bounds for the cardinality of rational points near to planar curves are known M. Huxley gave the following estimate:

let $f \in C^{(3)}(a,b)$, $c^{-1} < |f''(x)| < c$ for all $x \in (a,b)$, where $c > 0$ is constant and $0 < \mu < 1$, let $H_f^\mu(Q) = \{(\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2 : \frac{p_1}{q} \in (a,b), 0 < q \leqslant Q, |f(\frac{p_1}{q}) - \frac{p_2}{q}| < q^{-\mu-1}\}$ then for any $\varepsilon > 0$ for all sufficiently large Q one has $|H_f^\mu(Q)| \leqslant Q^{2-\mu+\varepsilon}$.

Further Beresnevich V.V. proved [2] the theorem giving the complementary lower bound.

Let I_0 be an open interval, $f \in C^3(I_0)$ and $f(x) \neq 0$ for all $x \in I_0$. Let $S \subset \mathbb{N}$ be an infinite subset and $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function satisfying

$$\lim_{t \to +\infty, t \in S} \psi(t) = \lim_{t \to +\infty, t \in S} \frac{1}{t\psi(t)} = 0$$

Let $B_f(Q, \psi, I) = \{(\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2 : \frac{p_1}{q} \in I, 0 < q \leqslant Q, |f(\frac{p_1}{q}) - \frac{p_2}{q}| < \frac{\psi(Q)}{Q}\}$ then for any interval $I \subset I_0$ there is a constant $c_1 > 0$ such that one has

$$|B_f(Q, \psi, I)| \geqslant c_1 Q^2 \psi(q) |I|$$

for all sufficiently large $Q \in S$. In particular, if I_0 is finite then taking $\psi(Q) = Q^{-\mu}$ with $0 < \mu < 1$ gives $|H_f^\mu(Q)| \geqslant c_1 Q^{2-\mu} |I_0|$.

Here generalizing for the 3-dimensions Euclidean space one has a more difficult and interesting problem. In particular, one can estimate the cardinality of rational points near a surface $z = xy$, where $z : \mathbb{R}^2 \to \mathbb{R}$.

Let $\varepsilon > 0,\ \delta > 0,\ I = I_1 \times I_2 \subset \diagdown^2,\ p_1,\ p_2,\ p_3,\ q \in \mathbb{Z},\ 0 < q \leqslant Q$.

Let

$$I_1 = \{x \in \mathbb{R} : |x - \frac{p_1}{q}| < \frac{Q^{-\frac{4}{3}+\varepsilon}}{\delta}\},$$

$$I_2 = \{y \in \mathbb{R} : |x - \frac{p_2}{q}| < \frac{Q^{-\frac{4}{3}+\varepsilon}}{\delta}\},$$

$$A(Q, I, \delta) = \{(\frac{p_1}{q}, \frac{p_2}{q}, \frac{p_3}{q}) \in \mathbb{Q}^3 : |\frac{p_1 p_2}{q^2} - \frac{p_3}{q}| < \frac{2}{\delta} H^{-\frac{4}{3}-2\varepsilon}\}.$$

Then $|A(Q, I, \delta)| \geqslant \frac{\delta^2}{2} Q^{\frac{8}{3}-2\varepsilon} |I_1||I_2|$.

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ON SOME ARITHMETIC PROPERTIES OF CONSTANTS RELATED TO THETA FUNCTIONS

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Many results about arithmetic properties of the elliptic functions [1,2] can be translated into the language of theta functions [3]. Here we present some of them.

We denote by $\xi_i \quad (i = 2, 3, 4)$ the approximating algebraic numbers; by n_i and L_i the degree and the length of these numbers, $\quad n = \deg \mathbb{Q}(\xi_2, \xi_3, \xi_4), \quad L = 1 + \frac{\ln L_2}{n_2} + \frac{\ln L_3}{n_3} + \frac{\ln L_4}{n_4}$.

We will use the following notation for theta functions

$$\theta_2 = 2q^{1/4} \sum_{n \geq 0} q^{n(n+1)}, \quad \theta_3 = 1 + 2 \sum_{n \geq 1} q^{n^2},$$

$$\theta_4 = 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}, \quad q = \exp(\pi i \tau).$$

Theorem 1. *Let $T = n(L \min(n_2, n_3, n_4) + \ln n)$. Then*

$$\sum_{i=2}^{4} |\pi\theta_i - \xi_i| > \exp(-\Lambda n T \ln T),$$

where Λ is some effective constant.

This result is a consequence of Theorem 1 [4] (for $\omega_1 = 1$) and the relations between the theta functions and elliptic functions [3].

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JOINT APPROXIMATION OF ZERO BY VALUES OF INTEGER POLYNOMIALS IN $\mathbb{R}^2 \times \mathbb{C}^2$

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Suppose that $P_n x = a_n x^n + \ldots + a_1 x + a_0$ is a polynomial with integer coefficients a_j, $H = \max_{1 \le j \le n} |a_j|$ is the height of $P(x)$.
In 1965 V.G. Sprindzhuk established a hypothesis, according to which one the system of inequalities

$$\begin{cases} |P_n(x)| < H^{-w} \\ |P_n(x)| < H^{-w} \end{cases} \tag{1}$$

has at $w > \frac{n-1}{2}$ only a finite number of solutions for almost all $(x, y) \in \mathbb{R}$. This hypothesis was proved in [1]. In [2] a generalization of this result for this approximations in different metrics was obtained, the theorem with an arbitrary monotonically decreasing function $\varphi(H)$ in the right-hand side of (1) and with some condition on convergence of a number bound with $\varphi(H)$ was proved in [3]. We proved the theorem extending the basic result in [3].

Theorem 1. *Let*

$$\lambda_j \ge -1,\ \mu_j \ge 0,\ 1 \le j \le 4,\ \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 = n - 6,$$
$$\mu_1 + \mu_2 + 2\mu_3 + 2\mu_4 = 1,$$

assume that the function $\psi(H)$ monotonically decreases, and $\sum_{H=1}^{\infty} \psi(H) < \infty$. Then the system of inequalities

$$\begin{cases} |P_n(x_1)| < H^{-\lambda_1} \psi^{\mu_1}(H) \\ |P_n(x_2)| < H^{-\lambda_2} \psi^{\mu_2}(H) \\ |P_n(z_1)| < H^{-\lambda_3} \psi^{\mu_3}(H) \\ |P_n(z_2)| < H^{-\lambda_4} \psi^{\mu_4}(H) \end{cases}$$

has only a finite number of solutions for almost all $(x_1, x_2, z_1, z_2) \in R^2 \times C^2$.

DIOPHANTINE APPROXIMATION WITH RESPECT TO DIFFERENT VALUATIONS

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Let $P_n = P_n(y) = a_n y^n + \cdots + a_1 y + a_0 \in \mathbb{Z}[y]$, $\deg P_n = n$ and $H = H(P_n) = \max_{0 \le i \le n} |a_i|$. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing function and $\sum_{n=1}^{\infty} \psi(n) < \infty$. Let $p \ge 2$ be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers, $|\cdot|_p$ be the p-adic valuation.

V.Sprindžuk (1965) proved Mahler's problem for P_n in the fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$. Later some generalizations of the convergence part of the Khintchine theorem (1924) were obtained for polynomials P_n. V.Bernik (1989), D.Vasiliyev (1998) and E.Kovalevskaya [1] proved results of this type for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$ respectively. We prove an analogue of the convergence part of the Khintchine theorem for simultaneous approximation of zero in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ by values of polynomials P_n. We notice that the problem under consideration belongs to the metric theory of Diophantine approximation of dependent values.

Further, we define a measure in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ as a direct product of the Lebesgue measures in $\mathbb{R}$, $\mathbb{C}$ and the Haar measure in $\mathbb{Q}_p$. We consider the system of inequalities

$$\begin{cases} |P_n(x)| < H^{\lambda_1} \psi^{\nu_1}(H), \\ |P_n(z)|^2 < H^{2\lambda_2} \psi^{2\nu_2}(H), \\ |P_n(\omega)|_p < H^{\lambda_3} \psi^{\nu_3}(H), \end{cases} \tag{1}$$

where $(x, z, \omega) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$, $\lambda_i \le 1$ $(i = 1, 2)$, $\lambda_3 \le 0$, $\lambda_1 + 2\lambda_2 + \lambda_3 = n - 3$, $\nu_i \ge 0$ $(i = 1, 2, 3)$, $\nu_1 + 2\nu_2 + \nu_3 = 1$, $\lambda_i - \nu_i < 1$ $(i = 1, 2)$, $\lambda_3 - \nu_3 < 0$.

Theorem 1. *The system of inequalities (1) is satisfied by at most finitely many polynomials $P_n \in \mathbb{Z}[y]$ for almost all $(x, z, \omega) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$.*

In order prove the theorem we develop Sprindžuk's method of essential and inessential domains and use a proof scheme from [2].

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ON CARDINALITY MEASURE OF THE SET OF POLYNOMIALS WITH BOUNDED VALUE OF DISCRIMINANTS

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Relationship between the value of an integer polynomial at a given point and the distance between this point and the nearest root of the polynomial is of great importance in the metric theory of transcendente numbers. If the derivative of the polynomial at the root is small then this distance can be considerable. Then we have to estimate the number of polynomials with a small derivative if we want to obtain an exact result (see [1]).

Let $P(x) \in \mathbb{Z}[x]$, $degP \leqslant n$, $H = H(P)$ be the hight of this polynomial. Let us denote by $F(Q)$ for some $p > 0$ and $\mu > 0$ the class of polynomials for which there exists a point $x \in \mathbb{R}$, such that the following inequalities are satisfied

$$\begin{cases} |P(x)| < H^{-\mu} \\ |P'(x)| < H^{1-p},\ H(P) \leqslant Q \end{cases}$$

Baker's result [2] was that

$$|F| \ll Q^{n+1-p},\ p < 1,$$

while we can improve it for $p < \frac{1}{2}$, obtaining that

$$|F| \ll Q^{n+1-2p},\ p < 1/2,$$

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ON SOME PROBLEM OF V.G. SPRINDJUK

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In the paper [1] V.G. Sprindjuk has posed a problem related to the analytic characterization of Dirichlet L-functions in the class of Dirichlet series admitting an analytic continuation to the complex plane as integral functions of first order. This problem has not been solved up to now and can be formulated in the following way:

Suppose a Dirichlet series

$$f(s) = \sum_{1}^{\infty} \frac{a_n}{n^s}, \quad s = \sigma + it \tag{1}$$

can be continued to an integral function on the complex plane and the function $f(s)$ satisfies the condition of growth of absolute value

$$|f(s)| < ce^{|s|\ln|s| + A|s|} \tag{2}$$

on the left half-plane $\sigma < 0$, where A is some constant. Furthermore assume that the values of this function at zeroes of Riemann's zeta-function do not increase too fast in the critical strip, more precisely, there exists $\eta > 0$ such that

$$\sum_{\rho} |f(\rho)| e^{-\tau|\rho|} = 0(\tau^{-n})$$

as $\tau \to 0$, where ρ runs over all nontrivial zeros of zeta-function.

It is required to prove that in this case the function $f(s)$ must be a Dirichlet L-function.

The authors obtained the following analytic criterion of a Dirichlet L-function in the class of Dirichlet series. This criterion can be useful in the solution of V.G. Sprindjuk's problem.

Theorem 1. *For a Dirichlet series of the form ([1]) with finite-valued, fully multiplicative coefficients the following conditions are equivalent:*

1) *a_n are periodic, and consequently Dirichlet series ([1]) is a Dirichlet L-function;*
2) *on the half-plane $\sigma > 1$ the function $f(s)$ admits an approximation by Dirichlet polynomials*

$$T_{n_k}(s) = \sum_{n=1}^{n_k} \frac{c_n}{n^s}, \quad s = \sigma + it$$

with degree of approximation $0(\frac{1}{\rho^{n_k}})$, where $\rho > 1$.

Remark *The sequence of Dirichlet polynomials $\{T_{n_k}(s)\}$ converges uniformly on any bounded domain in the complex plane, i.e. it determines an analytic continuation of the function $f(s)$. Moreover, $f(s)$ satisfies condition (2).*

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SOME ANALYTIC PROPERTIES FOR L-FUNCTIONS OF ELLIPTIC CURVES

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Let E be an elliptic curve given by the equation $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, with discriminant $\Delta = -16(4a^3 + 27b^2)$. Suppose that $\Delta \neq 0$, then the curve E is non-singular. For each prime p denote by $\nu(p)$ the number of solutions of the congruence $y^2 \equiv x^3 + ax + b (\bmod p)$, and let $\lambda(p) = p - \nu(p)$. H. Hasse proved that $|\lambda(p)| < 2\sqrt{p}$. Moreover, he and H. Weil attached to the curve E the L-function defined by the following Euler product

$$L_E(s) = \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod_{p \mid \Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1}, \qquad s = \sigma + it.$$

The latter product converges absolutely for $\sigma > 3/2$. Recently, the Hasse conjecture was proved, therefore the function $L_E(s)$ has analytic continuation to an entire function. Also, by the Shimura–Taniyama conjecture which was partially proved by A. Wiles, and in full form by C. Breuil, B. Conrad,F. Diamond and R. Taylor, the function $L_E(s)$ is the L-function attached to certain newform of weight 2 of some Hecke subgroup.

In the report we consider the universality of the function $L_E^k(s)$, where $k \neq 0$ is an integer number (the case of negative k uses an analog of RH for $L_E(s)$), and the functional independence of $L_E^k(s)$. For example, for $k > 0$ we have the following statement. Let K be a compact subset of the strip $\{s \in \mathbb{C} : \ 1 < \sigma < 3/2\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{ meas } \left\{\tau \in [0, T] : \ |L^k(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

[1] Partially supported by grant from Lithuanian Foundation of Studies and Sciences

The case $k = 1$ has been obtained in [1].

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SOME GENERALIZATIONS OF UNIFORMLY DISTRIBUTED SEQUENCES AND THEIR APPLICATIONS

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Let (X, d) be a compact metric space, P be a probability measure on the σ-algebra of Borel subsets of X. A sequence $x_1, x_2, \ldots \in X$ is called P-uniformly distributed if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} u(x_i) = \int_X u(x)P(dx)$$

for any continuous real function u. Thus, such a sequence represents the measure well in the certain sense. However, in applications we often need a finite set that represents the measure better than any other set with the same power. This leads to the following problem: find a set

$$Y_n^* = \operatorname*{arg\,min}_{Y=\{x_1,\ldots,x_n\}} \phi(Y)$$

for various forms of the criterion ϕ. For instance, in the theory of quadrature formulas and classification theory respectively

$$\phi(Y) = \sup_{u\in K}\left|\frac{1}{n}\sum_{i=1}^{n} u(x_i) - \int_X u(x)P(dx)\right|, \quad \phi(Y) = \int_X d(x, Y)P(dx)$$

are used, where K is some function class and $d(x, Y) = \inf_{y\in Y} d(x, y)$. We propose some non-traditional criteria. Sometimes the optimal sets Y_n^* are segments of the same sequence for all n. In particular, one meets such a situation in some problems for non-Archimedean metric spaces. We give various examples of applications to database analysis, mathematical sociology, and other. Some aspects of computer realization are discussed.

ON THE APPROXIMATION BY DESCRETE BIORTHOGONAL SERIES

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Let $\mu(n)$, $n \in \mathbb{N}$, be Möbius function, $\chi_1(\nu, k)$, $\chi_2(\nu, k)$, $\nu \in \mathbb{Z}$, $k \in \mathbb{N}$, be real primitive Dirichlet residue characters modulo k; $f(t)$ be 1-periodic continuous function.

We make a conversion of Fourier series

$$f(t) \sim \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m t}$$

into a series with descrete coefficients:

$$f(t) - c_0 \sim \sum_{n=1}^{\infty} s_n(f, t; \chi_1, \chi_2), \tag{1}$$

where

$$s_n(f, t; \chi_1, \chi_2) = I_n(f; \chi_1)g_n(t; \chi_1) + iI_n(f; \chi_2)h_n(t; \chi_2),$$

$$I_n(f; \chi_{1(2)}) = \frac{1}{n\tau(\chi_{1(2)})}\sum_{\nu=1}^{kn} \chi_{1(2)} f(\frac{\nu}{kn}),$$

$$\tau(\chi_{1(2)}) = \sum_{r=1}^{k} \chi(r) e^{2\pi i \frac{r}{k}},$$

$$g_n(t;\chi_1) = \sum_{d|n} \mu(\frac{n}{d})\chi_1(\frac{n}{d}) \cos 2d\pi t$$

$$h_n(t;\chi_2) = \sum_{d|n} \mu(\frac{n}{d})\chi_2(\frac{n}{d}) \sin 2d\pi t, n \in \mathbb{N}.$$

Sufficient conditions of convergence of the series (1) are obtained.

Theorem 1. *Let*

$$\sum_{m=-\infty}^{+\infty} \rho_m |c_m| < \infty,$$

where

$$\rho_m = 2^{1+\delta} \frac{\ln em}{\ln \ln em}, \ \ \delta > 0.$$

Then

(1) *the series (1) is absolutely and uniformly convergent;*
(2) *the estimate*

$$|f(t) - c_0 - \sum_{n=1}^{k} s_n(f,t;\chi_1,\chi_2)| \ll \int_0^{\frac{1}{k}} \varepsilon^{\frac{3}{2}} \rho(\frac{1}{\varepsilon}) \omega(f;\varepsilon) d\varepsilon$$

is true, where $\omega(f;\varepsilon)$ is modulus of continuity.

ESTIMATION OF THE TOTAL NUMBER OF THE RATIONAL POINTS ON A SET OF CURVES IN SPECIAL CASES

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Let p be a prime, k and l be positive divisors of $p-1$, $h = (p-1)/k$, $r = (p-1)/l$. For $n = h, r$ we denote

$$\mu_n = \{x \in Z_p | \quad x^n = 1\}$$

and M_n is a set of distinct coset representatives of μ_n in Z_p^*.

Under the influence of the paper [1] the author proved the inequality

$$\sum_{(u,v)\in U} |\{(x,y) \in \mu_h \times \mu_r | \quad ux - vy = 1\}| \quad \ll \quad (hrT^2)^{1/3}$$

for an arbitrary set $U \subset M_h \times M_r$, if $T = |U|$ satisfies the conditions $h^2r^2T < p^3$ and $T(\min(h,r))^2 > \max(h,r)$.

In the particular case $k = l$ such assertion was proved and applied for estimating of Gauss sums by D.R. Heath-Brown and S.V. Konyagin.

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ON A LINEAR THEOREM OF SYLVESTER

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Let a, b be positive integers, $(a, b) = 1$. J.J. Sylvester [1] proved that the greatest integer which can't be represented as $ax + by$ with nonnegative integers x, y is

$$ab - a - b \quad .$$

This result may be easily generalized as follows. Let $a_1, \ldots, a_n$ be pairwise coprime positive integers, $n \geq 2$, $A = a_1 \ldots a_n$, $A_i = A/a_i$. Then the greatest integer which isn't represented as $A_1x_1 + \ldots + A_nx_n$ with nonnegative integers $x_1, \ldots, x_n$ is equal to

$$(n-1)A - A_1 - \ldots - A_n \quad .$$

For $n = 3$ it was given as a problem at a student competition in mathematics. From here it follows that any positive integer which can't be represented as $A_1x_1 + \ldots + A_nx_n$ with nonnegative integers $x_1, \ldots, x_n$ is represented in a form

$$kA - k_1A_1 - \ldots - k_nA_n \quad ,$$

where $k, k_1, \ldots, k_n$ are positive integers, $k \leq n - 1$, and the inverse assertion is also true.

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ON THE REPRESENTATION OF PRIMES BY POLYNOMIALS (A SURVEY OF SOME RECENT RESULTS)

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About five years ago J. Friedlander and H. Iwaniec [1, 2] proved that there are infinitely many primes of the form $x^2 + y^4$. Inspired by their work, but by a totally different method, D.R. Heath-Brown [3] shows that the binary cubic form $x^3 + 2y^3$ represents infinitely many prime numbers, thereby confirming the conjecture of G.H. Hardy and J.E. Littlewood on the infinity of primes expressible as a sum of three cubes. Subsequently it has been shown [4] that any irreducble primitive binary cubic form with integral rational coefficients takes infinitely many prime values, if it takes at least one odd value. Indeed, we prove [5] an analogous theorem even for certain binary non-homogeneuos cubic polynomials. I intend to survey some of the ideas behind the proof of these results.

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ON UPPER ESTIMATION OF THE MEASURE OF A RATIONAL NUMBER SET GENERATED BY SCHMIDT CURVE

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Let $I = [a; b]$ be an interval, $f_1(x),\ f_2(x) \in C^3[a; b]$ and the curvature of the curve $\Gamma_2 = (f_1(x), f_2(x))$ be nonzero almost everywhere. In 1964 V. Schmidt [1] proved that for any $\varepsilon > 0$ the inequality

$$|F_2(x)| = |a_2f_2(x) + a_1f_1(x) + a_0| < H^{-2-\varepsilon}, \tag{1}$$

$$H = \max_{0 \leqslant j \leqslant 2} |a_j|$$

has infinitely many solutions only for the set of zero measure.

In [2] the inequality (1) was generalized for curves $\Gamma_n = (f_1(x), \dots, f_n(x))$ with the right-hand side of the inequality given as $H^{-n+1}\Psi(H)$.

Now for a monotone decreasing function $\Psi(H)$ everything depends on convergence or divergence of $\sum_{H=1}^{\infty} \Psi(H)$. In [3] they put H^{-w} as the right-hand side and for $w > 2$ found the exact value of Hausdorff measure for the set of solutions of the obtained inequality.

Nowadays a generalization of the theorem from [3] for any n is of great interest. One of the possible ways to obtain it is to find the exact estimate of the measure of those x, for which $F_n(x) < Q^{-w},\ w > n$ is true. This estimate holds at least for one collection $(a_0, a_1, \dots, a_n)$ and is obtained in [4] for $n < w < \frac{n}{4n^2+2n-4}$. If $n = 2$, then $2 < w < 2\frac{1}{8}$. It will be proved in the report, that $\frac{1}{8}$ can be replaced by $\frac{1}{2}$.

References

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MULTI-DIMENTIONAL DIOPHANTINE APPROXIMATIONS

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A brief survey on classical and recent results dealing with the general laws of Diophantine approximations will be presented in the lecture. The following topics are going to be considered.

(1) Evaluation of some Diophantine constants;
(2) Exponents of growth for the best Diophantine approximations;
(3) Phenomena of the degenerate dimension of subspaces generated by the best Diophantine approximations;
(4) Distribution of the directions of approximations;
(5) Vectors with a given Diophantine type.

ON POLYNOMIALS WITH SMALL DENOMINATORS

Pereverzeva N.A. (Grodno, Belarus)

Let $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be a polynomial with integer coefficients and

$$H = H(P) = \max_{0 \leqslant j \leqslant n} |a_j|$$

be it's height. The subject of Diophantine approximations is studying of sets $M_n(\omega)$ of $x \in \mathbb{R}$ such that the inequality $|P_n(x)| < H^{-\omega}$ has infinitely many solutions in polynomials $P(x)$ where ω is fixed. For fixed x the exponent ω characterizes the measure transcendence of x. If $|P'(x)| << H$, then all metric problems concerning $M_n(\omega)$ can be easily solved. But as $|P'(x)|$ decreases, it becomes necessary to estimate the frequency of appearance of the polynomials with a small derivative. Let $\mathcal{F}_n(Q)$ denote the class of polynomials $P_n(x)$ for which there is at least one point x, at which the following system of inequalities holds

$$\begin{cases} |P_n(x)| < Q^{-\omega},\ \omega > 0 \\ |P_n'(x)| < Q^{-\upsilon}, \\ H(P) \leqslant Q. \end{cases}$$

To solve the Baker-Schmidt problem about Hausdorff dimension of the set $M_n(\omega)$, $\omega > n$ the estimate $\#\mathcal{F}_n(Q) << Q^{n+1-0,1\upsilon}$ was obtained (see [1]). In [2], the equality $\#\mathcal{F}_n(Q) << Q^{n+1-\upsilon}$ was proved for $0 \leqslant \upsilon < 1$. Earlier it was found out that for $0 < \upsilon < n/2$ one has $\#\mathcal{F}_n(Q) << Q^{n+1-0,5\upsilon}$.

Theorem 1. *For $n \leqslant 5$ and $0 < \upsilon < \frac{5}{2}$ the following inequality is satisfied*

$$\#\mathcal{F}_n(Q) << Q^{6-0,6\upsilon}.$$

References

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ON THE EQUATION $1^k + 2^k + \ldots + x^k = y^z$

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We give a survey talk on the power values of power sums including some recent results obtained by Bennett, Győry, Jacobso, Walsh and the speaker.

SMALL DENOMINATORS IN MULTIPOINT PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

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Let $\lambda_j(k), k\in\mathbb{Z}^p, j = 1, \ldots, n$, be the roots of equation $\lambda^n + \sum\limits_{j=0}^{n-1} A_{n-j}(k)\lambda^j = 0$,

$$A_j(k) = \sum_{|s|\leq N_j} A_j^s k_1^{s_1} \ldots k_p^{s_p}, \quad A_j^s \in \mathbb{C}, \quad N_j \in \mathbb{N}, \quad j = 1, \ldots, n; \tag{1}$$

$C(n, m)$, $1 \leq m \leq n$, be the set of all $\omega = (i_1, \ldots, i_m) \in \mathbb{Z}^m$, where $1 \leq i_1 < \ldots < i_m \leq n$; $\Lambda_\omega(k) = \sum\limits_{j=1}^{m} \lambda_{i_j}(k)$, where $\omega = (i_1, \ldots, i_m)\in C(n, m)$.

Investigating a solvability of multipoint problems for partial differential equations such small denominators arise:

$$\Delta(k) = \det \left\| \lambda_q^{q_j-1}(k) \exp(\lambda_q(k)t_j) \right\|_{q_j=\overline{1,r_j};j=\overline{1,l}}^{q=\overline{1,n}},$$

$$0 \leq t_1 < \ldots < t_l \leq T,\ k \in \mathbb{Z}^p,$$

$$\Gamma_\omega(k) = \prod_{m\geq j>q\geq 1} (\lambda_{i_j}(k) - \lambda_{i_q}(k)),\ \omega = (i_1, \ldots, i_m) \in C(n, m),\ k \in \mathbb{Z}^p,$$

$$P_j(k) = \prod_{\omega \in C(\rho_j, r_j), \omega \neq \omega_j} \left(\Lambda_{\omega_j}(k) - \Lambda_\omega(k)\right),\ j = 1, \ldots, l-1,\ k \in \mathbb{Z}^p,$$

where $l \leq n$, $r_1+\ldots+r_l = n$, $\omega_j = (\rho_{j+1}+1, \rho_{j+1}+2, \ldots, \rho_{j+1}+r_j) \in C(\rho_j, r_j)$, $\rho_j = r_j+\ldots+r_l$, $j = 1, \ldots, l-1$.

Let $\Pi_N(\rho) = \{\vec{z} \in \mathbb{C}^N : \max\limits_{1\leq j\leq N} |z_j| \leq \rho\}$, $\rho > 0$, $\vec{Y} \equiv (\vec{y}_1, \ldots, \vec{y}_p) \in \mathbb{C}^{np}$, where $\vec{y}_q = (A_1^{s_{1,q}}, \ldots, A_n^{s_{n,q}}) \in \mathbb{C}^n$, $q = 1, \ldots, p$, $s_{j,q} = (0, \ldots, 0, N_j, 0, \ldots, 0)$ is multiindex of the length p, on the q-th position of which N_j is present; $\vec{U}$ is a vector formed by coefficients A_j^s in (1), which are not a components of the vector $\vec{Y}$.

For all fixed vectors $\vec{U}$ the next propositions are hold, where constants θ_j, ψ_j, ω, δ, γ are determined in explicit form by $p, n, r_1, \ldots, r_l, \rho, N_1, \ldots, N_n, T$.

Theorem 1. *For almost all (with respect to the Lebesgue measure in $\mathbb{C}^{np}$) vectors $\vec{Y} \in \Pi_{np}(\rho)$ the inequalities*

$$|\Gamma_{\omega_j}(k)| \geq |k|^{-\theta_j-\varepsilon},\ j = 1, \ldots, l,$$

$$|P_j(k)| \geq |k|^{-\psi_j-\varepsilon},\ j = 1, \ldots, l-1,\ \varepsilon > 0,$$

are satisfied for all (with the except of finite number) vectors $k \in \mathbb{Z}^p$.

Theorem 2. *For almost all (with respect to the Lebesgue measure in $\mathbb{C}^{np}$) vectors $\vec{Y} \in \Pi_{np}(\rho)$ and for almost all (with respect to the Lebesgue measure in $\mathbb{R}^l$) vectors $\vec{t} = (t_1, \ldots, t_l) \in [0, T]^l$ the inequality*

$$|\Delta(k)| > (1 + |k|)^{-\omega} \exp(-\delta |k|^\gamma)$$

is satisfied for all (with the except of finite number) vectors $k \in \mathbb{Z}^p$.

SPECTRUM OF ADELIC VLADIMIROV OPERATOR

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The group of adeles is of great interest in number theory. It can be used, for example, to understand better the nature of Riemann zeta.

An adele $x \in \mathbb{A}$ is a sequence $x = (x_\infty, x_2, x_3, x_5, ..., x_p, ...) = (x_\nu)$, where the ∞ symbol stands for the usual modulus, the primes p stand for non-archimedean p-adic valuations of $\mathbb{Q}$, and each x_ν belongs to the corresponding completion $\mathbb{Q}_\nu$, $\mathbb{Q}_\infty = \mathbb{R}$.

Define expression $|x|^\alpha$ with an infinite multi-index $\alpha = (\alpha_\infty, \alpha_2, \alpha_3, ..., \alpha_p, ...)$, $\alpha_\nu \in \mathbb{R}$, as the infinite product

$$|\xi|^\alpha = |\xi_\infty|_\infty^{\alpha_\infty} \cdot |\xi_2|_2^{\alpha_2} \cdot |\xi_3|_3^{\alpha_3} \cdot \ldots \cdot |\xi_p|_p^{\alpha_p} \cdot \ldots .$$

Theorem 1. *Let $\alpha_p \log p \to 0$ as $p \to \infty$, $\alpha_\nu > -1/q$ for all ν, $1 \le q < +\infty$. Then $|\xi|^\alpha$ represents a function on $\mathbb{A}$ which is finite almost everywhere, $|\xi|^\alpha \in L_q^{\mathrm{loc}}(\mathbb{A})$.*

The standard Fourier transformation defined on the space $S(\mathbb{A})$ of Schwartz–Bruhat functions can be continued onto $L_2(\mathbb{A}) \supset S(\mathbb{A})$, where it is a linear isometry.

Note. The space $S(\mathbb{A})$ of Schwartz–Bruhat functions is just a tensor product $S(\mathbb{A}) = S(\mathbb{R}) \otimes S(\mathbb{A}_0)$ of the Schwartz space on $\mathbb{R}$ and the Schwartz–Bruhat space on the group of finite adeles (i.e. adeles without the archimedean component x_∞). The product is complete with respect to Grothendieck topology. The space $S(\mathbb{A}_0)$ is an inductive limit of finite dimensional spaces.

Definition. Let $\alpha_p \log p \to 0$, $\alpha_\nu > -1/2$. *The pseudodifferential Vladimirov operator* V_α in $L_2(\mathbb{A})$ with domain $D(V_\alpha) = S(\mathbb{A})$ is defined as

$$V_\alpha \psi = \mathcal{F}^{-1}[|\xi|^\alpha (\mathcal{F}\psi)(\xi)].$$

Theorem 2. *The operator V_α with the domain $D(V_\alpha)$ is essentially self-adjoint. Its closure (also defined as V_α again) with the domain*

$$D(V^\alpha) = \{\psi \in L_2(\mathbb{A}) : |\xi|^\alpha (\mathcal{F}\psi)(\xi) \in L_2(\mathbb{A})\}$$

is self-adjoint and its spectrum is $[0, +\infty)$.

COLOURING THE ERDÖS UNIT-DISTANCE GRAPHS

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In our talk, we shall discuss the following classical combinatorial question going back to P. Erdös and H. Hadwiger (1940 - 1950): what is the minimum number of colours needed to paint the entire Euclidean space $\mathbb{R}^n$ in a way such that any two points at the distance exactly equal to one have distinct colours? In other words, what is the chromatic number $\chi(\mathbb{R}^n)$ of Euclidean space defined as the chromatic number of the infinite graph (the so called Erdös unit-distance graph) whose vertex set is $\mathbb{R}^n$ and whose edge set is formed by all pairs $x, y \in \mathbb{R}^n$ such that $|x - y| = 1$?

Various results have been obtained in connection with the above-mentioned question. For instance, it is known that $4 \le \chi(\mathbb{R}^2) \le 7$ and that $(1.239... + o(1))^n \le \chi(\mathbb{R}^n) \le (3 + o(1))^n$. The asymptotic upper bound is due to D.G. Larman and C.A. Rogers (1972) and the corresponding lower one was discovered by the author in 2000. The problem of improving the latter lower bound seems to be very hard. One of the most far-reaching approaches for solving it is in considering some special finite subgraphs of the unit-distance graph and in estimating their chromatic numbers from below with the help of linear algebra and number theory techniques. Recently we succeeded in developing a new method for doing that in the case when the vertex set of such a finite graph has a prescribed arithmetic coordinate structure.

In the talk, we shall first give a historical overview of the problem. Then we shall proceed to exhibiting our new results, and finally we shall discuss some possible extensions and applications of our method.

DIOPHANTINE APPROXIMATIONS IN THE FIELD OF REAL AND COMPLEX NUMBERS AND HAUSDORFF DIMENSION

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Let $\mathcal{L}_1(\omega)$ be the set of real numbers for which inequalities

$$|\alpha - p/q| < q^{-\omega_1 - 1} \text{ or } |\alpha q - p| < q^{-\omega_1}$$

have infinitely many solutions in integer p and natural q. Yarnik and Besikovitch found out that the Hausdorff dimension of $\mathcal{L}_1(\omega)$ equals $\frac{2}{\omega_1+1}$ when $\omega \geqslant 1$. This result was generalized for polynomials of arbitrary degree.

Let $\mathcal{L}_n(\omega)$ denote the set of $x \in \mathbb{R}$ such that the following inequality has infinitely many solutions in integer polynomials $P(x) \subset \mathbb{Z}[x]$:

$$|P_n(x)| = |a_n x^n + \cdots + a_1 x + a_0| < H^{-\omega_n},\ H = \max_{0 \leqslant j \leqslant n} |a_j|$$

In [1] a lower estimate for $\dim \mathcal{L}_n(\omega_n)$ when $\omega > n$ is obtained, and V. Bernik obtained an upper estimate (see [2]). Based on their works we can conclude that $\dim \mathcal{L}_n(\omega_n) = \frac{n+1}{\omega+1}$ if $\omega_n > n$.

Earlier the generalization was obtained for the case of complex numbers.

The following generalization of the mentioned results for simultaneous approximations in $\mathbb{R} \times \mathbb{C}$ has been proved.

Let $S_n(\omega)$ denote the set $(x, z) \in \mathbb{R} \times \mathbb{C}$, such that the system of inequalities

$$\max(|P_n(x)|, |P_n(z)|) < H^{-\omega}$$

has infinitely many solutions in $P_n(t) \in \mathbb{Z}[t]$.

Theorem 1. *There exists a constant c that doesn't depend on n such that if $n \geqslant 3$ and $\omega > \frac{n-2}{3}$ then*

$$\dim S_n(\omega) < c\frac{n+1}{\omega+1}$$

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DIOPHANTINE APPROXIMATIONS ON AN n-DIMENSIONAL SPHERE

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Let $S_n = \{x \in \mathbb{R}^n |\ |x| = 1\}$. Consider the system

$$\begin{cases} \|x_1 q\| < q^{-v} \\ \|x_2 q\| < q^{-v} \\ \quad \cdots \\ \|x_n q\| < q^{-v}, \end{cases} \tag{1}$$

where $(x_1, \cdots, x_n) \in S_n$ (here $\|t\|$ denotes the distance between t and the nearest integer).

Let $P_n(\upsilon)$ denote the set of all points on S_n for which (1) holds for infinitely many values of $q \in \mathbb{N}$. Similar problem on S_2 was considered in [1]. The value of Hausdorff dimension $\dim P_2(\upsilon)$ when $\upsilon > 1$ was proved to be equal to $\frac{1}{\upsilon+1}$ by M. Dodson and H. Dickinson.

For $\upsilon > 1$ one can prove the following lemma:

Lemma 1. *There exists a positive integer Q such that for any $q > Q$, $q \in \mathbb{N}$, from the inequalities*

$$\begin{cases} |x_1 - \frac{p_1}{q}| < q^{-1-\upsilon} \\ \quad \dots \\ |x_n - \frac{p_n}{q}| < q^{-1-\upsilon} \end{cases}$$

for a certain combination of $p_i \in \mathbb{Z}$ and some $(x_1, x_2, \dots, x_n) \in S_n$ it follows that $(\frac{p_1}{q}, \frac{p_2}{q}, \dots, \frac{p_n}{q}) \in S_n$.

Let T_n be the set of all rational points $x = (\frac{p_1}{q}, \frac{p_2}{q}, \dots, \frac{p_n}{q}) \in S_n$, $p_i \in \mathbb{Z}$, $q \in \mathbb{N}$. For any $x \in T_n$ denote $N(x) = q$. Then the following lemma holds:

Lemma 2. *For any distinct $x,\ y \in T_n$ such that $N(x) \leqslant Q$ and $N(y) \leqslant Q$ we have*

$$|x - y| > \frac{1}{Q}$$

Lemma 1 and Lemma 2 allows us to construct a regular system of points, which leads to the following lower bound for the Hausdorff dimension

$$\dim P_n(\upsilon) \leqslant \frac{n-1}{\upsilon+1}.$$

The upper bound can be easily obtained by summation: $\dim P_n(\upsilon) \geqslant \frac{n-1}{\upsilon+1}$.

This proves the following theorem.

Theorem 1. *For $\upsilon > 1$ Hausdorff dimension of the set $P_n(\upsilon)$ equals* $\dim P_n(\upsilon) = \frac{n-1}{\upsilon+1}$.

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LINEAR DIOPHANTINE APPROXIMATIONS ON THE PLANE WITH A CONTINUOUS PARAMETER

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During last years as a result of wide application of metric theory in number geometry (works of F. Getze) the following problem took on special significance.

Let $I = [a, b]$ be an interval, $Q > 0$ be big and $\delta > 0$ be small rational numbers. Traditional Diophantine approximations theory tries to clear up the question of solvability in integers q for systems of inequalities $max(\|\alpha_1 q\|, \|\alpha_2 q\|) < Q^{-\mu}$, $1 \leqslant q \leqslant Q$, where $\mu > 0$, $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\|x\|$ is the distance between $x \in \mathbb{R}$ and the nearest integer. It's a well known fact that if $\mu = \frac{1}{2}$ then the system is solvable for any (α_1, α_2). If right-hand member is diminishing then not all pairs (α_1, α_2) satisfy the system. This situation generates a need of distinguishing and estimating of these sets. μA is Lebesgue measure of a measurable set, c is a constant, that doesn't depend on δ or Q.

Theorem 1. *Let α_1 and α_2 be such that starting with some $q > q_0(\upsilon)$, $1 < \upsilon < 2$ an inequality $\|\frac{\alpha_2}{\alpha_1} q\| > q^{-\upsilon}$ holds. Let $\mathfrak{L}(\delta, Q)$ denote the set of $t \in I$ for which the system of inequalities*

$$\begin{cases} \max(\|\alpha_1 t)q\|, \|\alpha_2 t q\|) < \delta Q^{-1/2} \\ 1 \leqslant q < \delta Q \end{cases}$$

is solvable. Then

$$\mu \mathfrak{L}(\delta, Q) \leqslant c\delta^2 \mu I.$$

ON RECURRENCE IN THE AVERAGE

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Let X be a metric space with a metric $d(\cdot,\cdot)$ and a Borel sigma–algebra of measurable sets Φ. Let T be a measure preserving transformation of the measure space (X, Φ, μ) and let us assume that the measure of X is equal to 1. The well–known Poincare theorem asserts that for every point $x \in X$:

$$\forall \varepsilon > 0 \ \forall K > 0 \ \exists t > K : d(T^t x, x) < \varepsilon.$$

Let H_α be an ordinary Hausdorff measure on X. The following theorems 1 and 2 were proved by M. Boshernitzan. (A similar result was obtained independently by N.G. Moshchevitin).

Theorem 1. *Let X be a metric space with $H_\alpha(X) = C < \infty$ and T be a measure preserving transformation of (X, Φ, μ) into (X, Φ, μ).*
Then for almost every $x \in X$ $\liminf_{n\to\infty}\{n^\beta \cdot d(T^n x, x)\} < \infty$, where $\beta = 1/\alpha$.

We shall say that a measure μ is congruent to a measure H_h, if any μ–measurable set is H_α–measurable.

Theorem 2. *Let X be a metric space, H_α and μ are congruent and for any μ-measurable set A $\mu(A) = H_\alpha(A)$. Let T be a measure preserving transformation of X.*
Then for almost every $x \in X$ $\liminf_{n\to\infty}\{n^\beta \cdot d(T^n x, x)\} \leq 1$, where $\beta = 1/\alpha$.

In this work we obtain the mean value of the local recurrence and the N-recurrence constants and also the value of the recurrence constant in the topological case. We also apply this approach to the theory of continued fractions. Our result is the following.

Theorem 3. *For any $\varepsilon > 0$ and for almost all (with respect to Lebesgue measure) numbers $\alpha = [a_1, a_2, \dots]$ there exists an increasing sequence $\{n_\nu\}_{\nu\in\mathbf{N}}$, such that*

$$a_1 = a_{n_\nu+1}, \ a_2 = a_{n_\nu+2}, \ \dots \ , \ a_{k_\nu} = a_{n_\nu+k_\nu} \tag{1}$$

and $k_\nu \geq (6\ln 2/\pi^2 - \varepsilon) \cdot \ln n_\nu$.
2) For any $\delta > 0$ measure of those α for which there exists an increasing sequence $\{n_\nu\}_{\nu\in\mathbf{N}}$, such that (1) holds and $k_\nu \geq (1+\delta)/\ln 2 \cdot \ln n_\nu$ equals zero.

HARMONIC ANALYSIS ON TOTALLY DISCONNECTED GROUPS AND IRREGULARITIES OF POINT DISTRIBUTIONS

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We study point distributions in the multi-dimensional unit cube which possess the structure of finite abelian groups with respect to certain p-ary arithmetic operations. Such distributions can be thought of as finite subgroups in a compact totally disconnected group of the Cantor type. We apply methods of L^q harmonic analysis to estimate very precisely the L^q-discrepancies for such distributions. Following this approach, we explicitly construct point distributions with the minimal order of L^q-discrepancy for each q, $1 < q < \infty$.

ESTIMATES OF THE MEASURES OF SETS WHERE THE MODULUS OF A SMOOTH FUNCTION IS AN UPPER BOUND

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For a function f defined in $[0,T]$ we denote $G(f,\varepsilon) = \{t \in [0,T] : |f(t)| \le \varepsilon\}$, $\varepsilon > 0$. Let $\mu(A)$ be the Lebesgue measure in $\mathbb{R}$ of a measurable set $A \subset \mathbb{R}$; $C^n([0,T];\mathbb{R})$ (respectively $C^n([0,T];\mathbb{C})$) be the set of all real functions $f : [0,T] \to \mathbb{R}$ (respectively of all complex functions $f : [0,T] \to \mathbb{C}$), which have in $[0,T]$ the continuous derivatives of the degree $\le n$.

In the metric theory of diophantine approximations on real manifolds [1], in mathematical physics when investigating the problem of small denominators [2] lemma by A.S. Piartly [3] has numerous applications. In this lemma it is proved that for a function $f \in C^n([0,T];\mathbb{R})$, which satisfies the condition $|f^{(n)}(t)| \ge \delta$, $\delta > 0$, $t \in (0,T)$, the inequality $\mu(G(f,\varepsilon)) \le C_1(\varepsilon/\delta)^{1/n}$, $C_1 = C_1(n)$ holds.

The next propositions are generalizations of this Piartly result.

Theorem 1. *Let $f \in C^n([0,T];\mathbb{R})$, $p_j \in C^n([0,T];\mathbb{R})$, $j = 1,\dots,n$. If the inequality*

$$\left|f^{(n)}(t) + p_1(t)f^{(n-1)}(t) + \ldots + p_n(t)f(t)\right| \ge \delta, \quad \delta > 0, \quad t \in (0,T),$$

holds, then for all $\varepsilon > 0$

$$\mu(G(f,\varepsilon)) \le C_2 M^{\xi_n} \exp(\eta_n M T)(\varepsilon/\delta)^{1/n}, \quad C_2 = C_2(n,T),$$

where $M = 1 + \max\limits_{1\le j\le n} \|p_j(t)\|_{C^n[0,T]}$, and the constants ξ_n, η_n are determined by n.

Theorem 2. *Let $L(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$, where $a_j \in \mathbb{C}$, $j = 1,\dots,n$. Let λ_j, $j = 1,\dots,n$, be the roots of the polinomial $L(\lambda)$, $\Lambda = 1 + \max\limits_{1\le j\le n} |\lambda_j|$, $\Lambda^- = \min\limits_{1\le j\le n} Re\,\lambda_j$, $\psi = \max\limits_{t\in[0,T]} \exp(-\Lambda^- t)$. If the function $f \in C^n([0,T];\mathbb{R}(\mathbb{C}))$ is the solution of the Cauchy problem*

$$L(d/dt)f(t) = 0,\ f^{(j-1)}(0) = f_j,\ j = 1,\dots,n,\ |f_1| + \ldots + |f_n| > 0,$$

then for all ε, $0 < \varepsilon < \dfrac{g_f}{n2^n e^T \psi \Lambda^n}$, $g_f \equiv \max\limits_{1\le j\le n} \left\{|f_j|\Lambda^{-j}\right\}$,

$$\mu(G(f,\varepsilon)) \le C_3 \Lambda \left(\varepsilon\psi/g_f\right)^{1/(n-1)}, \quad C_3 = C_3(n,T).$$

Another analogues of Piartly's lemma are also proved.

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ON A LOCAL-GLOBAL PRINCIPLE IN NUMBER FIELD TOWERS FOR THUE CURVES

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This talk concerns the problem of finding algebraic integer points on an affine curve given by the equation

$$F(x, y) = a, \tag{1}$$

where $F(X, Y) \in \mathcal{O}_K[X, Y]$ is a homogeneous polynomial of degree n with coefficients in the ring of integers of a number field K, $a \in \mathcal{O}_K$.

Using a local-global method of B. Birch and results of K. Győry and Y. Bugeaud, we obtain effective bounds c_1 and c_2 in the following

Theorem 1. *Suppose that a is in the ideal generated by the coefficients of F. There are an algebraic extension L of K of degree $[L : K] \leq c_1$ and a solution $x, y \in \mathcal{O}_L$ to (1) such that* $\max(h(x), h(y)) \leq c_2$, *where $h(\alpha)$ denotes the height of an algebraic number α.*

CONVERGENCE CASE OF A KHINTCHINE-TYPE THEOREM FOR ANALYTIC FUNCTIONS WITH LARGE DERIVATIVES

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Let $\Psi(x)$ be a positive nonincreasing function in a real variable x. The classical result of Khintchine states that for almost all x (in the sense of Lebesgue measure over $\mathbb{R}$) inequality

$$\left| x - \frac{p}{q} \right| < \frac{\Psi(x)}{q} \tag{1}$$

has infinite or finite number of solutions in $(p, q) \in \mathbb{Z}\times\mathbb{N}$ depending respectively on the divergence or convergence of the series $\sum_{q=1}^{\infty} \Psi(q)$.

We establish the following

Theorem 1. *Let $\Psi(x)$ be a positive nonincreasing function in a real variable x defined over all $x > 0$ such that $\sum_{H=1}^{\infty} \Psi(H) < \infty$. Let $f_1(z), \ldots, f_n(z)$ be fixed analytic functions of complex variables over a domain D. We put $F(z) = a_0 + a_1 f_1(z) + \ldots + a_n f_n(z)$ where $a_0, \ldots, a_n$ are integers and $H(F) = \max\limits_{0\leq i\leq n} |a_i|$. Then the system of inequalities*

$$\begin{cases} |F(z)| & < & H(F)^{-\frac{n-2}{2}} \Psi^{1/2}(H(F)) \\ |F'(z)| & > & H(F)^{1/2} \end{cases}$$

has infinitely many solutions in functions $F(z)$ for the set of Lebesgue measure zero.

ALGEBRAIC INTERPOLATION

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Let us have an arbitrarily given function $f : [a, b] \to \mathbb{R}$ and the interpolation step $h = \frac{b-a}{N} > 0$. Then in an arbitrary manner we determine the function on the interval $(b, +\infty)$. The values $f(x_0)$, $f(x_1)$, ..., $f(x_k)$, ..., where $x_k = a + kh$, are independent variables of the general interpolation problem. Therefore we can consider countably dimensional $\mathbb{R}$-module V with a basis

$$f^k = f(x_k),\ k = \overline{0, \infty},$$

at the same time V is a ring with a natural product of formal series ($f^i \cdot f^j = f^{i+j}$). This ring contains the so-called finite differences

$$\Delta^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^j.$$

Theorem 1. *The set of vectors $\Delta^0, \Delta^1, \ldots, \delta^k, \ldots$ is a basis of space V and*

$$\Delta^m \cdot \Delta^n = \Delta^{m+n},$$

then we have, that the polynomial

$$P(x) = \sum_{i=0}^{n} \frac{\Delta^i}{h^i i!}(x - x_0)(x - x_1)\ldots(x - x_{i-1})$$

interpolates the function $f(x)$ at nodes x_k, $k = \overline{0, N}$.

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LEVEQUE'S U-POINTS ON SMOOTH CURVES

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We shall denote by H_p the height of a polynomial $P \in \mathbb{Z}[x]$ where the height of a polynomial with integer coefficients is defined as the maximum of the absolute values of its coefficients.

Definition. We shall say that a real number α is a Leveque's number of degree not greater than n if $\forall k \in \aleph\ \exists P_k \in \mathbb{Z}[x]|\ 0 < |P_k(\alpha)| < \frac{1}{H_{P_k}^k}$, $\deg P_k \leqslant n$. We shall denote by S_n the set of all Leveque's U-numbers of degree not greater than n. Let us put by definition $S_0 = \emptyset$.

A Leveque's U-number of degree n is an element of the set $U_n = S_n \backslash S_{n-1}$, $n \in \aleph$.

We shall say that a point $(x, y) \in \mathbb{R}^2$ is a U_n-point if $x, y \in U_n$.

One can note that U_1 is the same as the set of well known Liouville's numbers. It is obvious that

$$\sum_{n=1}^{\infty} 10^{-n!} \in U_1.$$

It is a little bit trickier to prove that

$$\sqrt[k]{\sum_{n=1}^{\infty} 10^{-n!}} \in U_1$$

(see [2]).

These examples show that U_k are not empty.

Erdös showed that each real number can be decomposed into a sum of two Liouville's numbers. Also each non-zero real number can be represented as a product of two Liouville's numbers.

One can represent this fact as follows: for any real number c there exists a U_n-point on the curves defined by $y = c - x$, $y = c/x$.

The theorem below is a direct generalization of this statement.

Theorem 1. *Let $f(x, y) = 0$ define a smooth curve in $\mathbb{R}^2$, say A. Suppose that A is neither a horizontal nor a vertical line, i.e.*

$$\exists\{(x_1, y_1), (x_2, y_2)\}\ |\ f(x_i, y_i) = 0,\ x_1 \neq\ x_2,\ y_1 \neq\ y_2$$

Then for any natural n there exists a U_n-point on A. Moreover, for any open

$$V \subset \mathbb{R}^2\ V \cap\ A \neq \emptyset \Rightarrow \exists\ U_n - point\ in\ V \cap\ A.$$

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