ON THE ETA TRANSFORMATION FORMULA PAPER OF J. ELSTRODT

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Abstract. We supply proofs of some of the results used in Elstrodt's paper [1].

The Dedekind $\eta(\tau)$ function is defined on the upper half-plane $\mathbb{H}$ in $\mathbb{C}$ by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}). \tag{1}$$

It is analytic on $\mathbb{H}$.

The aim is to prove

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau) \text{ for } \tau \in \mathbb{H},$$

where the branch of the function $z^{1/2}$ is chosen so that it is positive when z is real and positive.

We first consider the Weierstrass zeta function

$$\zeta(z, \tau) = \frac{1}{z} + \sum_{m,n \in \mathbb{Z}}{}' \left(\frac{1}{z + m\tau + n} - \frac{1}{m\tau + n} + \frac{z}{(m\tau + n)^2} \right) \tag{2}$$

where $\tau \in \mathbb{H}$, $z \in \mathbb{C} \backslash \Lambda$, and the primed summation means that $(0, 0)$ is excluded from the summation.

We now rewrite the formula for $\zeta(z, \tau)$ using the identities:

$$\pi \cot \pi x = \sum_{n=\infty}^{\infty} \frac{1}{x + n}$$

and

$$\sum_{k=-\infty}^{\infty} \frac{1}{(u + k)^2} = \pi^2 / \sin^2 \pi u.$$

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We break up the double sum in (2) according as $m = 0$ and $m \neq 0$:

$$S_1 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} + \frac{z}{n^2} \right) = \pi \cot \pi z - \frac{1}{z} + \frac{2z\pi^2}{6}. \tag{3}$$

$$\begin{aligned} S_2 &= \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{z+m\tau+n} - \frac{1}{m\tau+n} + \frac{z}{(m\tau+n)^2} \right) \\ &= \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (\pi \cot \pi(z+m\tau) - \pi \cot m\pi\tau + z\pi^2 \csc^2 \pi m\tau) \\ &= \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (\pi \cot \pi(z+m\tau) + z\pi^2 \csc^2 \pi m\tau). \qquad (4) \end{aligned}$$

Equations (2), (3) and (4) then give

$$\begin{aligned} \zeta(z,\tau) &= \frac{\pi^2}{3} z + \pi \cot \pi z + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (\pi \cot \pi(z+m\tau) + z\pi^2 \csc^2 \pi m\tau) \\ &= 2\pi^2 \left(\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\sin^2 n\pi\tau} \right) z + \\ &\quad + \pi(\cot \pi z + \sum_{n=1}^{\infty} (\cot \pi(z+n\tau) + \cot \pi(z-n\tau))). \end{aligned}$$

Evaluating $\zeta(z,\tau)$ for $z = \frac{1}{2}, z = \frac{\tau}{2}$ using $\cot(\frac{\pi}{2} + \theta) = -\tan\theta$ gives

$$\varphi_1(\tau) := \frac{i}{2\pi} \zeta\left(\frac{1}{2}, \tau\right) = \frac{\pi i}{12} + \frac{\pi i}{2} \sum_{n=1}^{\infty} \frac{1}{\sin^2 n\pi\tau} \tag{5}$$

$$\varphi_2(\tau) := \frac{i}{2\pi} \zeta\left(\frac{\tau}{2}, \tau\right) = \tau\phi_1(\tau) + \frac{1}{2} \text{ (\textit{Legendre's equation})}. \tag{6}$$

For

$$
\begin{aligned}
\frac{i}{2\pi}\zeta\left(\frac{1}{2},\tau\right) &= \frac{i}{2\pi}\left(2\pi^2(\frac{1}{6}+\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau})\right)\frac{1}{2} \\
&\quad + \frac{\pi i}{2\pi}(\cot\frac{\pi}{2}+\sum_{n=1}^{\infty}(\cot\pi(\frac{1}{2}+n\tau)+\cot\pi(\frac{1}{2}-n\tau))) \\
&= \frac{\pi i}{12}+\frac{\pi i}{2}\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}+\frac{i}{2}\sum_{n=1}^{\infty}(-\tan n\pi\tau+\tan n\pi\tau) \\
&= \frac{\pi i}{12}+\frac{\pi i}{2}\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{i}{2\pi}\zeta\left(\frac{\tau}{2},\tau\right) &= \frac{i}{2\pi}\left(2\pi^2(\frac{1}{6}+\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau})\right)\frac{\tau}{2} \\
&\quad + \frac{\pi i}{2\pi}(\cot\frac{\pi\tau}{2}+\sum_{n=1}^{\infty}(\cot\pi(\frac{\tau}{2}+n\tau)+\cot\pi(\frac{\tau}{2}-n\tau))) \\
&= \tau\left(\frac{i\pi}{12}+\frac{i\pi}{2}\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}\right)+\frac{i}{2}\lim_{N\to\infty}\cot(\frac{2N+1}{2})\pi\tau \\
&= \tau\left(\frac{i\pi}{12}+\frac{i\pi}{2}\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}\right)+\frac{1}{2},
\end{aligned}
$$

as $\lim_{N\to\infty}\cot(\frac{2N+1}{2})\pi\tau = -i$ follows from $\Im(\tau) > 0$.

But definition (2) implies

$$
\frac{1}{\tau}\varphi_1\left(-\frac{1}{\tau}\right) = \varphi_2(\tau). \tag{7}
$$

For

$$\begin{aligned}
\frac{2\pi}{i\tau}\varphi_1\left(-\frac{1}{\tau}\right) &= \frac{1}{\tau}\zeta\left(\frac{1}{2},-\frac{1}{\tau}\right)\\
&= \frac{1}{\tau}\left(2+\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{1}{\frac{1}{2}-\frac{m}{\tau}+n}-\frac{1}{-\frac{m}{\tau}+n}+\frac{1}{2(-\frac{m}{\tau}+n)^2}\right)\right)\\
&= \frac{2}{\tau}+\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{1}{\frac{\tau}{2}-m+\tau n}-\frac{1}{-m+n\tau}+\frac{1}{2\tau(-\frac{m}{\tau}+n)^2}\right)\\
&= \frac{2}{\tau}+\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{1}{\frac{\tau}{2}-m+\tau n}-\frac{1}{-m+n\tau}+\frac{\tau}{2(-m+\tau n)^2}\right)\\
&= \frac{2}{\tau}+\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{1}{\frac{\tau}{2}+m+\tau n}-\frac{1}{m+n\tau}+\frac{\tau}{2(m+\tau n)^2}\right)\\
&= \zeta\left(\frac{\tau}{2},\tau\right)\\
&= \frac{2\pi}{i}\varphi_2(\tau).
\end{aligned}$$

Hence from (6),

$$\frac{1}{\tau^2}\varphi_1\left(-\frac{1}{\tau}\right)=\varphi_1(\tau)+\frac{1}{2\tau}. \tag{8}$$

Then we get

$$\varphi_1(\tau)=\frac{\eta'}{\eta}(\tau). \tag{9}$$

To prove this, we need the following result:

LEMMA 0.1.

$$\frac{1}{sin^2 z}=-4\sum_{n=1}^{\infty}ne^{2inz}\ \ \textit{if}\ \Im(z)<0. \tag{10}$$

Proof.

$$\begin{aligned}
\frac{1}{sin^2 z} &= \frac{-4}{(e^{iz}-e^{-iz})^2}=\frac{-4e^{2iz}}{(1-e^{2iz})^2}=\frac{-4u}{(1-u)^2},\quad \text{where } u=e^{2iz},\\
&= -4\sum_{n=1}^{\infty}nu^n \text{ if } |u|<1\\
&= -4\sum_{n=1}^{\infty}ne^{2inz} \text{ if } \Im(z)>0
\end{aligned}$$

□

We now take the logarithmic derivative of (1):

$$\begin{aligned}
\frac{\eta'(\tau)}{\eta(\tau)} &= \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \frac{ne^{2\pi i n\tau}}{1-e^{2\pi i n\tau}} \\
&= \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} ne^{2\pi i n\tau} \sum_{m=0}^{\infty} e^{2\pi i n m\tau} \\
&= \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} ne^{2\pi i n m\tau} \\
&= \frac{\pi i}{12} - 2\pi i \sum_{m=1}^{\infty} \frac{1}{-4\sin^2 \pi\tau} \\
&= \frac{\pi i}{12} + \frac{\pi i}{2} \sum_{m=1}^{\infty} \frac{1}{\sin^2 \pi\tau} \\
&= \varphi_1(\tau) \quad \text{from (5).} \qquad (11)
\end{aligned}$$

Then (8) and (9) give

$$\begin{aligned}
\frac{1}{\tau^2}\frac{\eta'}{\eta}\left(-\frac{1}{\tau}\right) &= \frac{\eta'}{\eta}(\tau) + \frac{1}{2\tau} \\
\frac{d}{d\tau}\log\eta\left(\frac{-1}{\tau}\right) &= \frac{d}{d\tau}(\log(\tau) + \frac{1}{2}\log\tau) \\
\log\eta\left(\frac{-1}{\tau}\right) &= \log\eta(\tau) + \frac{1}{2}\log\tau + c.
\end{aligned}$$

Taking $\tau = i$ gives $c = -\frac{1}{2}\log i$.

Hence

$$\begin{aligned}
\log\eta\left(\frac{-1}{\tau}\right) &= \log\eta(\tau) + \frac{1}{2}\log\tau - \frac{1}{2}\log i \\
\eta\left(\frac{-1}{\tau}\right) &= \eta(\tau)e^{\frac{1}{2}\log\tau - \frac{1}{2}\log i} \\
&= \eta(\tau)e^{\frac{1}{2}(\log|\frac{\tau}{i}| + i\arg(\tau) - i\arg i)} \\
&= \eta(\tau)e^{\frac{1}{2}(\log|\frac{\tau}{i}| + i\arg(\frac{\tau}{i}))}, \quad \text{as } 0 < \arg\tau < \pi, \\
&= \eta(\tau)\sqrt{\frac{\tau}{i}}.
\end{aligned}$$

References

[1] J. Elstrodt, Manuscripta Mat. 121, 457-459, 2006